3 AMATH

3.1. AMATH 231

3.1.1. Topics

> Before you brush your teeth,
> parametrize your curves.
>
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- Vector Calculus
  - gradient vector field
  - Conservation in physics
  - line (path) integral

$$\int_C f\ ds = \int_a^b f(\mathbf{g}(t))\|\mathbf{g}'(t)\|\ dt$$

$$W = \int_C \mathbf{F}\cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t))\cdot\mathbf{g}'(t)dt$$

  - Path-independence and the Fundamental Theorems of Calculus for Line Integrals

$$\int_{C_{AB}} \mathbf{F}\cdot d\mathbf{x} = \int_{C_{AB}} \vec{\nabla} f\cdot d\mathbf{x} = f(B) - f(A)$$

  - First Fundamental Theorem for Line Integrals

$$f(\mathbf{x}) = \int_{x_0}^{x} \mathbf{F}\cdot d\mathbf{y} \implies \vec{\nabla} f(\mathbf{x}) = \mathbf{F}(\mathbf{x})$$

  - over closed curves

$$\oint_C \mathbf{F}\cdot d\mathbf{x}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t))\cdot\mathbf{g}'(t)dt = \oint f\ ds$$

where the scalar valued function $f(\mathbf{g}(t)) = \mathbf{F}(\mathbf{g}(t))\cdot\widehat{\mathbf{T}}(t)$

- Green's Theorem

$$\int_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{x} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \iint_{\mathcal{D}} (\nabla \times \mathbf{F})_z dA$$

- divergence

$$\text{div}\mathbf{F} = \vec{\nabla} \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- divergence of position vector

$$\vec{\nabla} \cdot \frac{1}{r^3}\mathbf{r} = 0 \qquad (x, y, z) \neq 0$$

- curl

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \vec{\nabla} \times \mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}$$

$$\nabla \times \mathbf{F} = \mathbf{0} \implies \mathbf{F} \text{ is gradient}$$

- vorticity
  $\mathbf{v}(x, y) = (v_1(x, y), v_2(x, y))$, then vorticity is

$$\Omega(x, y) = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

- Mean value theorem for (double) integrals

$$f(x^*, y^*) = \overline{f}_D = \frac{1}{A(D)} \iint_D f(x, y) dA$$

$$[\text{curl}\mathbf{F}(p_1, p_2)]_z = \Omega(p_1, p_2) = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \oint_{C_\varepsilon} \mathbf{F} \cdot d\mathbf{x}$$

- Total outward flux

$$\oint_C \mathbf{F} \cdot \widehat{\mathbf{N}} ds = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \widehat{\mathbf{N}}(t) ds = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \widehat{\mathbf{N}}(t) \|\mathbf{g}'(t)\| dt$$

- Divergence Theorem

$$\oint_C \mathbf{F} \cdot \widehat{\mathbf{N}} ds = \iint_D \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right] dA = \iint_D \vec{\nabla} \cdot \mathbf{F} \; dA$$

$$\vec{\nabla} \cdot \mathbf{F}(\mathbf{p}) = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \oint_{C_\varepsilon} \mathbf{F} \cdot \widehat{\mathbf{N}} ds$$

$$\vec{\nabla} \cdot \mathbf{F}(\mathbf{p}) = \lim_{\varepsilon \to 0} \frac{1}{\frac{4}{3}\pi\varepsilon^3} \iint_{S_\varepsilon} \mathbf{F} \cdot \widehat{\mathbf{N}} ds$$

- Circulation integrals/Outward flux integrals around singularities
- Surface integration
  * Surface parametrizations
  * Normal vectors to surfaces from parameterizations

$$\mathbf{N}(u_0, v_0) = \pm \mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0)$$

  where $\mathbf{T}_u = \frac{\partial \mathbf{g}(u,v)}{\partial u}$.

* surface integrals of scalar functions

$$\int_S f\ dS = \iint_{u,v} f(\mathbf{g}(u,v))\|\mathbf{N}(u,v)\|dudv$$

* surface area

$$S = \iint_{\mathcal{D}_{uv}} \left\|\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}\right\| dudv$$

* flux integral

$$\iint_S \mathbf{F}\cdot\widehat{\mathbf{N}}dS = \iint_{D_{uv}} \mathbf{F}(\mathbf{g}(u,v))\cdot\mathbf{N}(u,v)dA$$

* Gauss Divergence Theorem in $\mathbb{R}^3$

$$\underbrace{\iint_S \mathbf{F}\cdot\widehat{\mathbf{N}}dS}_{\text{surface integral}} = \underbrace{\iiint_D \text{div}\mathbf{F}\ dV}_{\text{volume integral}}$$

– Let $U(t)$ denote the total amount of substance X in region D at time t. $u(\mathbf{x},t)$ denote the concentration of substance X at a point $\mathbf{x}\in R\subset\mathbb{R}^3$.

$$U(t) = \iiint_D u(\mathbf{x},t)dV \implies U'(t) = \iiint_D \frac{\partial u}{\partial t}(\mathbf{x},t)dV$$

– The flow of X is defined by the flux density vector $\mathbf{j}(\mathbf{x},t)$. The net outward substance X through an infinitesimal element of surface area dS centered at point $x\in\partial D$ is given by $\mathbf{j}(\mathbf{x},t)\cdot\widehat{\mathbf{N}}(\mathbf{x})dS$. The total outward flux: $\iint_{\partial D}\mathbf{j}(\mathbf{x},t)\cdot\widehat{\mathbf{N}}(\mathbf{x})dS$. By Divergence Theorem, we have

$$\iint_{\partial D}\mathbf{j}(\mathbf{x},t)\cdot\widehat{\mathbf{N}}(\mathbf{x})dS = \iiint_D \vec{\nabla}\cdot\mathbf{j}(\mathbf{x},t)dV$$

and $U'(t) = -\iiint_D \vec{\nabla}\cdot\mathbf{j}(\mathbf{x},t)dV$, then **integral form of the conservation law** for substance X:

$$\iiint_D \left(\frac{\partial u}{\partial t}(\mathbf{x},t) + \vec{\nabla}\cdot\mathbf{j}(\mathbf{x},t)\right)dV = 0$$

– ∇ operator

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\frac{\partial u}{\partial v_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v_1} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v_1} = \nabla u\cdot\frac{\partial \mathbf{x}}{\partial v_1}$$

∇ in polar/cylindrical/spherical coordinates in $\mathbb{R}^3$

– Stoke's Theorem

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \iint_S (\nabla\times\mathbf{F})\cdot\widehat{\mathbf{N}}dS$$

– final calculation (using spherical polar coordinate is easiest)

$$\mathbf{F} = \frac{1}{r^n}\mathbf{r} = \frac{1}{(x^2+y^2+z^2)^{n/2}}[x\mathbf{i}+y\mathbf{j}+z\mathbf{k}] \implies \vec{\nabla}\cdot\mathbf{F} = \frac{3-n}{r^n}$$

- Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \qquad x \in [-\pi, \pi]$$

  - piecewise C^1
  - pointwise convergence for a Fourier series (2π-period and piecewise C^1)
  - even & odd extension of a function defined on $(0, \pi)$
  - $$\langle f_n, f_n \rangle = \frac{a_0^2}{2}\pi + \pi \sum_{k=1}^{n} [a_k^2 + b_k^2] \le \|f\|_2^2$$
  - $\{f_n\}$ converges pointwise to f: $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$, for all $x \in [a, b]$.
  - Series $\sum_{k=1}^{\infty} a_k$ in $C[a, b]$ converges uniformly to f means that the sequence $\{S_n\}$ of partial sums converges uniformly to f

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{\infty} a_k \right\|_{\infty} = 0$$

  - Series $\sum_{k=1}^{\infty} a_k$ in $C[a, b]$ converges in the mean to f means that the sequence $\{S_n\}$ of partial sums converges in the mean to f

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{\infty} a_k \right\|_{2} = 0$$

  - Weierstrass M-test (also covered in MATH 148): If $|a_n(x)| < M_n, \quad \forall x \in [a, b]$, $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} a_n(x)$ converges absolutely for each $x \in [a, b]$, with sum $f(x)$, it converges uniformly to $f(x)$ on $[a, b]$.
  - * f_p piecewise continuous $\implies$ converges in the mean to f_p on any finite interval
    * f_p piecewise C^1 $\implies$ converges pointwise to f_p for all $x \in \mathbb{R}$
    * f_p piecewise C^1 and continuous $\implies$ converges uniformly to f_p on any finite interval
  - Complex Fourier series of a τ-periodic function
  - Parseval's formula: $\langle f, f \rangle = \|f\|_2^2 = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)^2 dt = \tau \sum_{-\infty}^{\infty} |c_n|^2$
  - Fourier Transform:

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

3.1.2. Selected Proof

Conservation of Energy Assumptions: $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative.

Proof

▸

$$E(t) = \frac{1}{2} m \|\mathbf{v}(t)\|^2 + V(\mathbf{x}(t))$$

$$E'(t) = \frac{m}{2}\frac{d}{dt}[\mathbf{v}(t)\cdot\mathbf{v}(t)] + \frac{d}{dt}V(x_1(t),\ldots,x_n(t))$$

$$\frac{d}{dt}[\mathbf{v}(t)\cdot\mathbf{v}(t)] = \mathbf{v}'(t)\cdot\mathbf{v}(t) + \mathbf{v}(t)\cdot\mathbf{v}'(t) = 2\mathbf{a}(t)\cdot\mathbf{v}(t)$$

By chain rule:

$$\frac{d}{dt}V(x_1(t),\ldots,x_n(t)) = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} = \vec{\nabla}V\cdot\mathbf{v}$$

Put together,

$$E'(t) = \mathbf{v}(t)\cdot[m\mathbf{a}(t)] + \mathbf{v}(t)\cdot\vec{\nabla}V = \mathbf{v}(t)\cdot[m\mathbf{a}(t)+\vec{\nabla}V] = \mathbf{v}(t)\cdot[m\mathbf{a}(t)-\mathbf{F}(\mathbf{x}(t))] = \mathbf{v}(t)\cdot\mathbf{0} = 0$$

□

Work Integrals

$$\mathbf{F} = -\vec{\nabla}V$$

Proof

▸

$$W = \int_{C_{AB}}\mathbf{F}\cdot d\mathbf{x} = \int_{C_{AB}}\vec{\nabla}V\cdot d\mathbf{x} = -[V(B)-V(A)] = V(A)-V(B) = -\Delta V$$

□

Second Fundamental Theorem of Line Integrals Let $\mathbf{F}:\mathcal{U}\to\mathbb{R}^n$ be a continuous vector field on a connected open set $\mathcal{U}\subset\mathbb{R}^n$, and let $\mathbf{x}_1,\mathbf{x}_2$ be two points in $\mathcal{U}$. If $\mathbf{F} = \nabla f$, where $f:\mathcal{U}\to\mathbb{R}$ is a C^1 scalar field, and $\mathcal{C}$ is any curve in $\mathcal{U}$ joining $\mathbf{x}_1$ to $\mathbf{x}_2$, then

$$\int_{\mathcal{C}}\mathbf{F}\cdot d\mathbf{x} = f(\mathbf{x}_2) - f(\mathbf{x}_1)$$

Proof

▸ Let $\mathcal{C}$ be given by $\mathbf{x} = \mathbf{g}(t),\quad t_1\le t\le t_2$, so that $\mathbf{x}_1 = \mathbf{g}(t_1)$, $\mathbf{x}_2 = \mathbf{g}(t_2)$. Bt the hypothesis,

$$\begin{aligned}\int_{\mathcal{C}}\mathbf{F}\cdot d\mathbf{x} &= \int_{\mathcal{C}}(\nabla f)\cdot d\mathbf{x}\\ &= \int_{t_1}^{t_2}\nabla f(\mathbf{g}(t))\cdot\mathbf{g}'(t)dt \qquad \text{by definition of line integral}\\ &= \int_{t_1}^{t_2}\frac{d}{dt}[f(\mathbf{g}(t))]dt \qquad \text{Chain rule}\\ &= f(\mathbf{g}(t_2)) - f(\mathbf{g}(t_1)) \qquad \text{second FTC}\\ &= f(\mathbf{x}_2) - f(\mathbf{x}_1)\end{aligned}$$

□

Proposition 5.1

$$\begin{aligned}\|f\|_2^2 &= \int_a^b f(x)^2dx\\ &\le \max_{a\le x\le b}[f(x)]^2(b-a)\\ &= \left[\max_{a\le x\le b}|f(x)|\right]^2(b-a)\\ &= (b-a)\|f\|_\infty^2\end{aligned}$$

Proposition 5.2 If f_n converges uniformly in piecewise continuous $[a,b]$, then converges in (i) mean and (ii) pointwise. **Proof**
▸ For (i), use prop 5.1 and squeeze theorem. For (ii) by definition, $|f_n(x) - f(x)| \leq \|f_n - f\|_\infty$, then... □

3.2. AMATH 251

3.2.1. Topics

- First-order DEs
  - An equation relating an unknown function and one or more of its derivatives is called a differential equation.
  - The order of a differential equation is the order of the highest derivative that appears in it.
  - IVP & IC
  - Ordinary differential equations: the unknown function (dependent variable) depends on only a single independent variable.
  - If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are, the equation is called a partial differential equation.
  - general & particular solution
  - Slope fields & solution curves
  - **Theorem** Existence and Uniqueness of Solutions
    $\frac{dy}{dx} = f(x,y), \quad y(a) = b$. has only one solution is defined on I, if $f(x,y)$ & $\frac{\partial f}{\partial y}$ are continuous on some rectangle R in the xy-plane that contains the point (a,b) in its interior.
  - Separable Equations
  - Linear First-Order
    $$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0$$
    Integrating factor: $\rho(x) = e^{\int P(x)dx}$
  - **Theorem** Unique solution for linear first-order equation
    $P(x)$ and $Q(x)$ are continuous on the open interval I containing x_0
  - Substitution Methods
    * $\frac{dy}{dx} = F\left(\frac{y}{x}\right) \longrightarrow v = \frac{y}{x}$
    * Bernoulli Equation
    $$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad \longrightarrow v(x) = y^{1-n}$$
  - *Exactness (will not be tested on the final)*
    $$M(x,y) + N(x.y)\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \iff \text{exact in an open rectangle } R$$
  - Reducible Second-Order Equations