Proposition 5.2 If f_n converges uniformly in piecewise continuous $[a,b]$, then converges in (i) mean and (ii) pointwise. **Proof**

▸ For (i), use prop 5.1 and squeeze theorem. For (ii) by definition, $|f_n(x) - f(x)| \le \|f_n - f\|_\infty$, then... □

3.2. AMATH 251

3.2.1. Topics

- First-order DEs
  - An equation relating an unknown function and one or more of its derivatives is called a differential equation.
  - The order of a differential equation is the order of the highest derivative that appears in it.
  - IVP & IC
  - Ordinary differential equations: the unknown function (dependent variable) depends on only a single independent variable.
  - If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are, the equation is called a partial differential equation.
  - general & particular solution
  - Slope fields & solution curves
  - **Theorem** Existence and Uniqueness of Solutions
    $\frac{dy}{dx} = f(x,y), \quad y(a) = b$. has only one solution is defined on I, if $f(x,y)$ & $\frac{\partial f}{\partial y}$ are continuous on some rectangle R in the xy-plane that contains the point (a,b) in its interior.
  - Separable Equations
  - Linear First-Order
    $$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0$$
    Integrating factor: $\rho(x) = e^{\int P(x)dx}$
  - **Theorem** Unique solution for linear first-order equation
    $P(x)$ and $Q(x)$ are continuous on the open interval I containing x_0
  - Substitution Methods
    * $\frac{dy}{dx} = F\left(\frac{y}{x}\right) \longrightarrow v = \frac{y}{x}$
    * Bernoulli Equation
    $$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad \longrightarrow v(x) = y^{1-n}$$
  - *Exactness (will not be tested on the final)*
    $$M(x,y) + N(x.y)\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \iff \text{exact in an open rectangle } R$$
  - Reducible Second-Order Equations

- Dependent variable y missing
- Independent variable x missing

- **Models in Chapter 1**
  - Natural Growth and Decay $\frac{dx}{dt} = kx$
  - Newton's Cooling $\frac{dT}{dt} = k[A(t) - T]$
  - Torricelli's Law
    Suppose that a water tank has a hole with area a at its bottom, from which water is leaking. Denote by $y(t)$ the depth of water in the tank at time t, and by $V(t)$ the volume of water in the tank then. It is plausible – and true, under ideal conditions – that the velocity of water exiting through the hole is $v = \sqrt{2gy}$, which is the velocity a drop of water would acquire in falling freely from the surface of the water to the hole. As a consequence,

$$\frac{dV}{dt} = -av = -a\sqrt{2gy} \longrightarrow \frac{dV}{dt} = -k\sqrt{y} \quad \text{where } k = a\sqrt{2g}$$

    Alternatively, Let $A(y)$ denote the horizontal cross-sectional area of the tank at height y.

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = A(y)\frac{dy}{dt} \implies A(y)\frac{dy}{dt} = -a\sqrt{2gy} = -k\sqrt{y}$$

  - Mixture Problem:

    $\frac{dp}{dt}$ = Rate of change of p in time = rate pollution in − rate pollution out
    = (rate water in)(concentration pollutions in) − (rate water out) (concentration pollution out)

- Mathematical Models and Numerical Methods
  - Population Models:
    $\beta(t)/\delta(t)$ - # of births/deaths per unit of per population per unit time at time t

$$\frac{dP}{dt} = [\beta(t) - \delta(t)]\,P$$

    - Logistic equation $\frac{dP}{dt} = kP(M - P), \qquad P(0) = P_0$

$$\implies P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}} \qquad \lim_{t\to+\infty} P(t) = \frac{MP_0}{P_0 + 0} = M$$

      M: limiting population / carrying capacity
    - A constant solution of a differential equation is sometimes called an **equilibrium solution**.
    - the critical point c is **stable** if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x_0 - c| < \delta \implies |x(t) - c| < \varepsilon$$

      for all $t > 0$. Otherwise it is unstable.
    - Logistic Population with Harvest $\frac{dx}{dt} = kx(M - x) - h$

- Acceleration-Velocity Models
  - Resistance Proportional to Velocity
    $F_R = -kv. \quad m\frac{dv}{dt} = -kv - mg. \qquad |v_\tau| = \frac{mg}{k}.$
  - Resistance Proportional to Square of Velocity
    $F_R = -kv|v|. \qquad m\frac{dv}{dt} = -mg - kv|v|. \qquad |v_\tau| = \overline{v} = \sqrt{\frac{mg}{k}}$
- Newton's Law of Gravitation: $F = \dfrac{GMm}{r^2}$

$$\left.\begin{aligned} \frac{dv}{dt} = \frac{d^2r}{dt^2} = -\frac{GM}{(R+y)^2} = -\frac{GM}{r^2} \\ \frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr} \end{aligned}\right\} \Longrightarrow v\frac{dv}{dr} = -\frac{GM}{r^2} \Longrightarrow v = \sqrt{v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right)}$$

  Consider the interval of existence, we must have the radicand > 0. Thus we can find the escape velocity $v = \sqrt{\dfrac{2GM}{R}}$.
- Numerical Approximation
  - $\dfrac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$ Step size h. $\qquad y_{n+1} = y_n + hf(x_n, y_n).$
  - Improved Euler Method

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ u_{n+1} &= y_n + h \cdot k_1 \qquad \text{predictor} \\ k_2 &= f(x_{n+1}, u_{n+1}) \\ y_{n+1} &= y + h \cdot \frac{1}{2}(k_1 + k_2) \qquad \text{corrector} \end{aligned}$$

- Dimensional Analysis
  - Two principles
    1. One can only add, subtract or equate physical quantities with the same physical dimensions.
    2. Quantities with different dimensions may be combined by multiplication with dimensions.
  - Dimensionless Variables
  - Buckingham-π Theorem

$$Q_n = f(Q_1, \ldots, Q_{n-1}) \qquad \text{is equivalent to} \qquad \pi_k = h(\pi_1, \ldots, \pi_{k-1})$$

    r independent fundamental physical dimensions. $k = n - r$.
  - Pendulum Model
- Linear Equations of Higher order
  - *boundary value problem* / initial value problem
  - **Theorem** Principle of Superposition for Homogeneous Equations: $y = c_1y_1 + c_2y_2$ is also a solution on I.
  - **Theorem** Existence and Uniqueness for Linear Equations: $y'' + p(x)y' + q(x)y = f(x)$ has unique solution on I that satisfies $y(a) = b_0, y'(a) = b_1$.

- homogeneous & nonhomogeneous (associated homogeneous)
- linear independence of functions
- Wronskian. Suppose the functions $f_1, \ldots, f_n$ are $n-1$ times differentiable on some interval I:

$$W(f_1, \ldots, f_n) = \det \begin{bmatrix} f_1 & \ldots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \ldots & f_n^{(n-1)} \end{bmatrix}$$

  $f_1, \ldots, f_n$ linearly independent $\implies W(f_1, \ldots, f_n) \equiv 0$ on I.
- **Theorem** General Solution for a Linear Homogeneous Equation.

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y = 0 \tag{3.1}$$

  Let $\phi(x)$ be any solution of (3.1), $y_1, \ldots, y_n$ be linearly independent solutions on I, then there exists $c_1, \ldots, c_n$ such that

$$\phi(x) = \sum_{i=1}^{n} c_i y_i(x), \qquad \forall x \in I$$

  Note the difference from Superposition Theorem... I got no marks on proving this in midterm...

  Proof ($n = 2$) Let $\phi(x)$ be a solution of (3.1) on I. Let $a \in I$. Consider the linear system.

$$(*) \qquad \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \phi(a) \\ \phi'(a) \end{pmatrix}$$

  Since y_1, y_2 are linearly independent on I , $W(y_1, y_2) \neq 0$ on I. Thus $\det(M) \neq 0$ and $(*)$ has a solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M^{-1} \begin{pmatrix} \phi(a) \\ \phi'(a) \end{pmatrix}$$

  Using these values of c_1, c_2 define

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

  Then $y(x)$ satisfies the IVP on I consisting of (3.1) and $y(a) = \phi(a), y'(a) = \phi'(a)$. But $\phi(a)$ also satisfies this IVP on I. So by E/U we must have:

$$\phi(x) = y(x) = c_1 y_1(x) + c_2 y_2(x) \quad x \in I$$

  In other words, given $y_1, \ldots, y_n$ linearly independent solutions of (3.1), and arbitrary constants $c_1, \ldots, c_n$

$$c_1 y_1(x) + \ldots + c_n y_n(x)$$

  is a general solution of (3.1). □
- General Solution for a Linear Non-Homogeneous Equation.
- Homogeneous, linear ODEs with constant coefficients
  * characteristic equation/polynomial: $a_n r^n + \ldots + a_1 r + a_0 = 0$
  * Three cases: (c_i are arbitrary constants)
    - Linear independence verification uses Wronskian.
    - Proofs of the last two involve differential operator D
    1. distinct real roots: $y = c_1 e^{r_1 x} + \ldots + c_n e^{r_n x}$

2. repeated real roots (multiplicity k): $e^{\bar{r}x}, xe^{\bar{r}x}, \dots, x^{k-1}e^{\bar{r}x}$
3. complex roots ($\alpha \pm i\beta$): $e^{\alpha x}\cos(\beta x), \quad e^{\alpha x}\sin(\beta x)$

2 & 3. Repeated complex roots:
$e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x), \dots, x^{k-1}e^{\alpha x}\cos(\beta x), x^{k-1}e^{\alpha x}\sin(\beta x)$

- Application
  - Mass spring damp: $mx'' + cx' + kx = 0$
  - pendulum: $s = l\theta, \qquad ml\theta'' = -mg\sin(\theta)$

  Two models are of the same form $y'' + b_1 y + b_0 y = 0$

$b_1 = 0$			simple harmonic motion
$b_1^2 - 4b_0 < 0$	two complex root	underdamped	oscillatory with amplitude decaying
$b_1^2 - 4b_0 = 0$	one real repeated root	critically damped	not oscillatory
$b_1^2 - 4b_0 > 0$	two real roots	overdamped	not oscillatory

- Non-homogeneous DE
  - Undetermined Coefficients
  - Variation of Parameters
  - Application
    - Forced, undamped motion: resonance and beating
    - Forced, damped motion: practical resonance

- Linear Systems of DEs
  - definition $\mathbf{x}' = \underbrace{P(t)}_{\substack{\text{coefficient}\\ \text{matrix}}} \mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$
  - E/U: $P(t), \mathbf{f}(t)$ are continuous on an open interval I containing point t_0, then there exists a unique solution on I.
  - Superposition
  - Wronskian of $\mathbf{x}_1, \dots, \mathbf{x}_n$ (which are solutions of $\mathbf{x}' = P(t)\mathbf{x}$) is

$$W(\mathbf{x}_1, \dots, \mathbf{x}_n) = \det(M) = \det\begin{bmatrix}\mathbf{x}_1(t) & \dots & \mathbf{x}_n(t)\end{bmatrix} = \det\begin{bmatrix} x_{11}(t) & \dots & x_{n1}(t) \\ \vdots & \ddots & \vdots \\ x_{1n}(t) & \dots & x_{nn}(t)\end{bmatrix}$$

  dependent, $W \equiv 0$; independent, $W \neq 0, \forall t \in I$.
  - General Solution of Homogeneous/Non-Homogeneous Linear Systems

$$\vec{x}' = P(t)\vec{x} \tag{3.2}$$

  Proof of Homogeneous one (responsible for final)

Proof
▸ Let $\vec{x}(t)$ be any solution on I of (3.2). Let $t_0 \in I$, and $M(t)$ be as in the definition of the Wronskian. Since $\vec{x}_1, \dots, \vec{x}_n$ are linearly independent on I,

$$\det(M(t_0)) = W(\vec{x}_1(t_0), \dots, \vec{x}_n(t_0)) \neq 0$$

Thus the linear system $M(t_0)\vec{c} = \vec{x}(t_0)$ $\qquad (*)$
has a unique solution

$$\vec{c} = M^{-1}(t_0)\vec{x}(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Define $\vec{y}(t) = c_1\vec{x}_1(t)+\ldots+c_n\vec{x}_n(t)$. This is a solution of (3.2) by the Superposition Principle and satisfies the initial condition $\vec{y}(t_0) = \vec{x}(t_0)$. But $\vec{x}(t)$ is also a solution of (3.2) satisfying the same IC. By the E/U Theorem we must have

$$\vec{x}(t) = \vec{y}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) \qquad \forall t \in I$$

□

- Eigenvalue Method [1]

$\lambda_1, \lambda_2 \in \mathbb{R}$	$\mathbf{v}_1, \mathbf{v}_2$	$c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$
λ	$\mathbf{v}, \mathbf{u}$	$c_1 e^{\lambda t}\mathbf{v} + c_2\left(e^{\lambda t}\mathbf{u} + t e^{\lambda t}\mathbf{v}\right)$
$\lambda_{1,2} = \alpha \pm i\beta$	$\mathbf{v}_{1,2} = \mathbf{u} \pm i\mathbf{w}$	$c_1 e^{\alpha t}\left(\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{w}\right) + c_2 e^{\alpha t}\left(\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{w}\right)$

- solution curves
  - ∗ saddle point: nonzero distinct eigenvalues of opposite sign

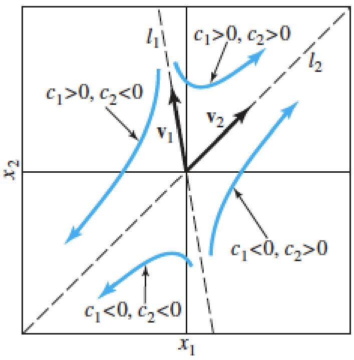

FIGURE 5.3.1. Solution curves $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when the eigenvalues λ_1, λ_2 of $\mathbf{A}$ are real with $\lambda_1 < 0 < \lambda_2$.

  - ∗ Nodes (sink): distinct negative eigenvalues. Origin: improper nodal sink

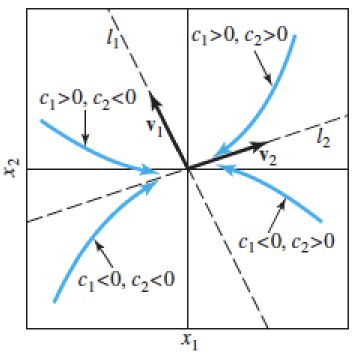

FIGURE 5.3.2. Solution curves $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when the eigenvalues λ_1, λ_2 of $\mathbf{A}$ are real with $\lambda_1 < \lambda_2 < 0$.

  - ∗ Nodes (source): distinct positive eigenvalues. Origin: improper nodal source

[1] $(A - \lambda I)\mathbf{u} = \mathbf{v}$, $\mathbf{u}$ is a generalized eigenvector of λ

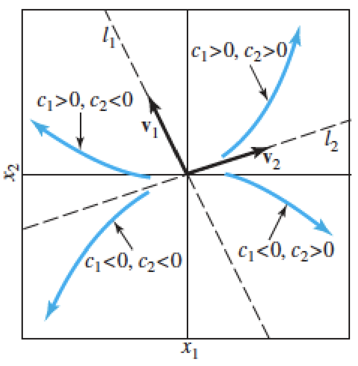

FIGURE 5.3.3. Solution curves $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when the eigenvalues λ_1, λ_2 of $\mathbf{A}$ are real with $0 < \lambda_2 < \lambda_1$.

* Repeated positive eigenvalue.
  - with two independent eigenvectors. Origin: proper nodal source

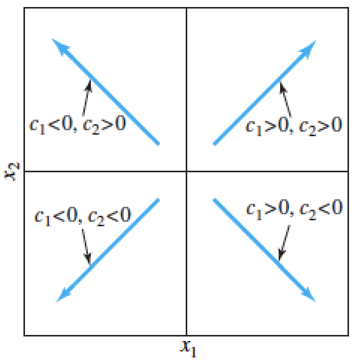

FIGURE 5.3.6. Solution curves $\mathbf{x}(t) = e^{\lambda t}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has one repeated positive eigenvalue and two linearly independent eigenvectors.

  - without two independent eigenvectors. Origin: improper nodal source

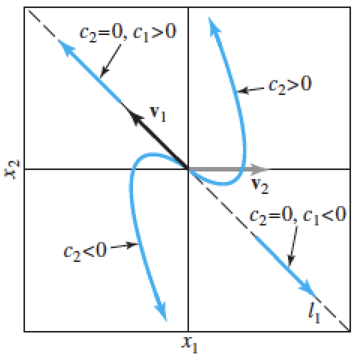

FIGURE 5.3.7. Solution curves $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda t} + c_2(\mathbf{v}_1t + \mathbf{v}_2)e^{\lambda t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has one repeated positive eigenvalue λ with associated eigenvector $\mathbf{v}_1$ and "generalized eigenvector" $\mathbf{v}_2$.

* Repeated negative eigenvalue.
  - with two independent eigenvectors. Origin: proper nodal sink (5.3.8)

- without two independent eigenvectors. Origin: improper nodal sink (5.3.9)

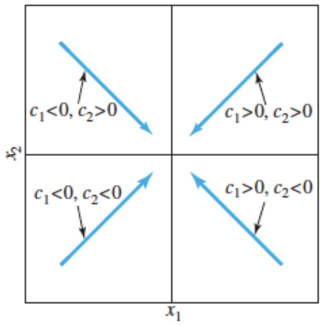

FIGURE 5.3.8. Solution curves $\mathbf{x}(t) = e^{\lambda t}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has one repeated negative eigenvalue λ and two linearly independent eigenvectors.

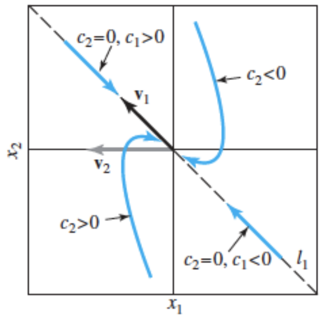

FIGURE 5.3.9. Solution curves $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has one repeated negative eigenvalue λ with associated eigenvector $\mathbf{v}_1$ and "generalized eigenvector" $\mathbf{v}_2$.

* Complex conjugate eigenvalues and eigenvectors
  - pure imaginary: center
  - negative real part: spiral sink
  - positive real part: spiral source

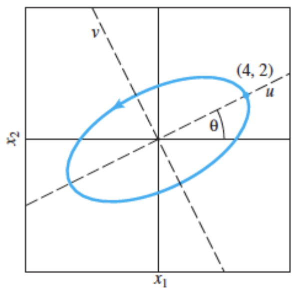

FIGURE 5.3.11. Solution curve $x_1(t) = 4\cos 10t - \sin 10t$, $x_2(t) = 2\cos 10t + 2\sin 10t$ for the initial value problem in Eq. (46).

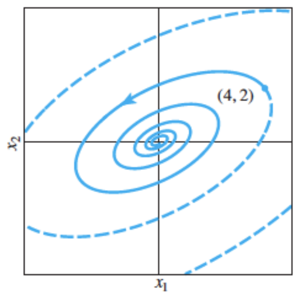

FIGURE 5.3.13. Solution curve $x_1(t) = e^{-t}(4\cos 10t - \sin 10t)$, $x_2(t) = e^{-t}(2\cos 10t + 2\sin 10t)$ for the initial value problem in Eq. (54). The dashed and solid portions of the curve correspond to negative and positive values of t, respectively.

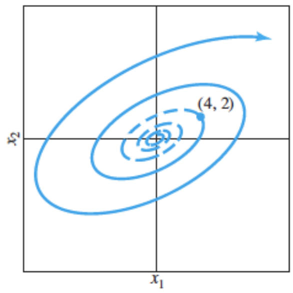

FIGURE 5.3.14. Solution curve $x_1(t) = e^{t}(4\cos 10t + \sin 10t)$, $x_2(t) = e^{t}(2\cos 10t - 2\sin 10t)$ for the initial value problem in Eq. (59). The dashed and solid portions of the curve correspond to negative and positive values of t, respectively.

– Fundamental Matrix: $\Phi(t) = \begin{bmatrix}\mathbf{x}_1(t) & \dots & \mathbf{x}_n(t)\end{bmatrix}$, where $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^n$ are n linearly independent solutions of $\mathbf{x}' = P(t)\mathbf{x}$ on I.
  - Propositions
    * Every solution $\mathbf{x}(t)$ can be written $\mathbf{x}(t) = \Phi(t)\mathbf{c}$ where $\mathbf{c} \in \mathbb{R}^n$.
    * invertible
    * $\Phi'(t) = P(t)\Phi(t)$
  - **Theorem** (Fundamental Matrix Solution)

$$\mathbf{x}' = P(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \qquad \text{unique solution is} \quad \mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0, t \in I$$

– Nonhomogeneous Linear Systems: Variation of Parameters. $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t)$

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = \Phi(t)\mathbf{c} + \Phi(t)\int \Phi(t)^{-1}\mathbf{f}(t)\,dt$$

- Laplace Transforms

- Definitions: $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$
- Unit step: $u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$
- exponential order: $|f(t)| \leq Me^{ct}, \qquad \text{for } t \geq T \qquad (2)$
- Existence of the Laplace Transform (responsible for final): If f is piecewise continuous on $t \geq 0$ and of exponential order as $t \to \infty$ with constant c in eq(2), then $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > c$.
  converge absolutely $\implies$ converges $\implies$ exists for $s > c$
  Proof
  ▸ Since f is piecewise continuous on $t \geq 0$, we can find $M \geq 0$ such that (2) is satisfied with $T = 0$. i.e.
  $$|f(t)| \leq Me^{ct}, \qquad \text{for } t \geq 0$$
  $\int_0^\infty Me^{ct}e^{-st}dt$ converges if $s > c$. Thus using a comparison theorem $\int_0^\infty |f(t)e^{-st}|dt$ converges for $s > c$.
  It follows that $\int_0^\infty f(t)e^{-st}dt$ converges. □
- Gamma function
- Uniqueness of the Inverse Laplace Transform
- Transform of Derivatives $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$
- Corollary
  $$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) = F(s)$$
- **Theorem** (Laplace Transform of Integrals) responsible for final
  If $f(t)$ is piecewise continuous on $t \geq 0$ and is of exponential order as $t \to \infty$ (with constants c, T, M) then
  $$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s} \quad \text{for } s > c$$
  equivalently:
  $$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$
  Proof
  ▸ Since f is piecewise continuous, on $t \geq 0$. $g(t) = \int_0^t f(t)dt$ is continuous on $t \geq 0$, g' is piecewise continuous on $t \geq 0$.
  Further,
  $$|g(t)| = \left|\int_0^t f(\tau)d\tau\right| \leq \int_0^t |f(\tau)|d\tau \leq \int_0^t Me^{c\tau}d\tau = \frac{M}{c}(e^{ct} - 1) \leq \frac{M}{c}e^{ct} \qquad t \geq 0$$
  So $g(t)$ is of exponential order as $t \to \infty$ and we can apply the Theorem on Laplace Transform of Derivatives.
  $$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{g'(t)\right\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\} \qquad \text{for } s > c$$
  $$\implies \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\} \qquad \text{for } s > c$$
  □

- Translation: $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a), \quad s > a+c$
- differentiation of transforms: $\mathcal{L}\{-tf(t)\} = F'(s)$
- convolution: $\mathcal{L}\{f(t) * g(t)\} = F(s) \cdot G(s)$
- translation: $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \qquad \text{for } s > c$

- Appendix
  - Method of Successive Approximations: $\dfrac{dy}{dx} = f(x,y)$, then $y_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f(t, y_{n-1}(t))dt$.
  - Existence for Linear Systems. The IVP has a solution on the entire interval I.

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(a) = \mathbf{b}$$

3.3. AMATH 390

3.3.1. Topics

This course provides an introduction to some of the deep connections between mathematics and music; mathematics will be used to provide insights into several important aspects of music. Topics covered include: modelling the acoustics of string, wind and percussion instruments with 1D and 2D partial differential equations, pitch and harmonics, frequency response and signal sampling with Fourier transforms, and the advantages and disadvantages of various scales and tuning systems (Pythagorean and just intonation, equal and well temperament).

3.4. AMATH 391