AMATH 251

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Contents

Info

0.1 Schedule change

- lectures
  - Mon/Wed MC4063
  - Thur RCH 103 10:30
- Office Hours MC 6132
  - Wednesday, 4:30-5:30 p.m.
  - Thursday 4:00-5:00 p.m.
- TUT Fri 3:30
  - Friday Sept.14 Maple MC 3006
  - After next week TUT MC 4063
- Tutor: M. C.[1]
  - MC6501
  - Office Hour: after Friday tut.

0.2 Maple

Quick note: File → New → Worksheet

[1] too harsh...

1 Intro

1.1 Overview

In this course, we'll use differential equations as mathematical models of physical systems.

Definition 1.1.1. A differential equations (DE) is a equation for an unknown function involving one or more derivatives of the equation.

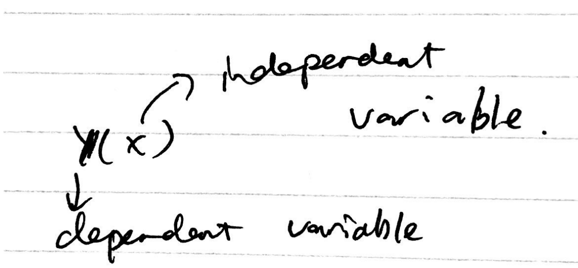
The unknown function is represented in terms of a dependent variable and the variables it depends on are called independent variable(s).

Simple Example

$$\frac{dy}{dx} = \cos(x)$$

unknown function $y(x)$

- $y \rightarrow$ dependent variable
- $x \rightarrow$ independent variable



Simple mathematical model Newton's 2^{nd} law: "the rate of change of momentum of an object is equal to the sum of the forces acting on the object".

Write this as a DE:

$$\frac{d(mv)}{dt} = F \tag{1.1}$$

- t - time
- m - mass of object

- v - velocity of object
- F - Forces acting

If the mass of the object is constant, this becomes

$$m\frac{dv}{dt} = F \tag{1.2}$$

If we know m and F, this can be thought of as a DE for the unknown $v(t)$, where v is dependent variable, t is independent variable.

Example An object is thrown vertically upward with speed v_0. If the mass of object is m, and the only force acting is gravity, find the velocity of the object at any later time.

Solution Assumptions

1. only force acting is gravity
2. mass is constant
3. Assume object is close enough to the surface of earth that force of gravity is constant.
4. take upward as positive direction

Model:

$$m\frac{dv}{dt} = -mg \tag{1.3}$$

$$v(0) = v_0 \tag{1.4}$$

where g is the gravitational acceleration.

Units:

- m - kg
- t - s
- v - m/s
- v_0 - m/s
- $g = 9.8m/s^2$

Equations (3) - (4) can be solved to find $v(t) = v_0 - gt$.

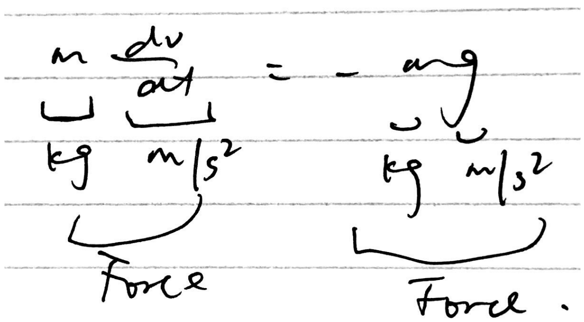
Definition 1.1.2. An initial condition (IC) is an equation which gives the value of the dependent variable for a specific value of the independent variable

Definition 1.1.3. An initial value problem (IVP) (abbr. learned from Jordan) is a differential equation together with one or more initial conditions.

Key thing for us - how to solve a given IVP.

When formulating mathematical models, it is important to be aware what physical quantities the variables represent.

$$\begin{aligned} \text{Model:} \qquad m\frac{dv}{dt} &= -mg \\ kg \times m/s^2 &= kg \cdot m/s^2 \\ \text{Force} &= \text{Force} \end{aligned}$$



Solution:

$$\begin{aligned} \underline{v(t)} &= \underline{v_0} - \underline{gt} \\ m/s &= m/s - (m/s^2) \times (s) \end{aligned}$$

1.2 Fundamental Questions in the study of DEs

1. Does a solution to a given IVP exist? (existence)
2. Is there more than one solution? (uniqueness)
3. How do we find/approximate the solution? (Method)

Classification of DEs: See extra supplementary materials.

Most general, n^{th} order ODE:

$$H(x, y, y', y'', \cdots, y^{(n)}) = 0 \qquad (1)$$

where H is a given function.
To solve (1) means to find the unknown function $y(x)$.

More formally,

Definition 1.2.1. The function $\phi(x)$ is a solution of (1) on the open interval $I = \{x \in \mathbb{R} : a < x < b\}$ if $\phi, \phi', \cdots, \phi^{(n)}$ exist on I and

$$H(x, y, y', y'', \cdots, y^{(n)}) = 0 \quad \text{for all } x \in I.$$

The interval I is called the interval of existence (or domain of definition) of the solution.

Example $y' = y^2$ It is easy to check that $\phi(x) = \dfrac{1}{1-x}$ stisfies this DE where it is defined, i.e. on $(-\infty, 1)$ and $(1, \infty)$
There are two solutions of the DE

$$y = \frac{1}{1-x}, \quad x \in (-\infty, 1)$$

$$y = \frac{1}{1-x}, \quad x \in (1, \infty)$$

2 First Order ODEs

2.1 Intro

Most general, first order ODE: $H(x, y, y') = 0$
If H is linear as y' this can be written

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \qquad \text{(2) (quasi-linear form)}$$

Let $D \subset \mathbb{R}^2$ be an open, connected, nonempty set. If $N(x, y) \neq 0$ on D then (2) can be written on D as

$$\frac{dy}{dx} = f(x, y) \qquad \text{(3)(Normal form or standard form)}$$

Example Solve the DE $\dfrac{dy}{dx} = \cos(x)$

Solution Integrate both sides of DE with respect to x

$$\int \frac{dy}{dx} = \int \cos(x)dx$$

$$y(x) = \sin(x) + C$$

So for any constant C the function $y(x) = \sin(x) + C$, $x \in \mathbb{R}$ is a solution of the DE.

Definition 2.1.1. A general solution of a first order ODE is a solution containing one arbitrary constant that represents almost all the solutions of the DE.

In previous example $y(x) = \sin(x) + C$ is a general solution of the DE.

Definition 2.1.2. A particular solution of a first order ODE is a solution that contans no arbitrary constant

When solving an IVP, we will look for the particular solution that satisfies the given initial condition.

Example Solve the IVP $\dfrac{dy}{dx} = \cos(x)$, $y(\pi) = 1$

Solution
General solution: $y(x) = \sin(x) + C$
Initial condition: $y(\pi) = 1 \implies 1 = \sin(\pi) + C \implies C = 1$
Solution of the IVP is $y(x) = \sin(x) + 1, \quad x \in \mathbb{R}$

Example Consider the IVP $xy' = y, \quad y(0) = 5$
It can be shown that a general solution for the DE is $y(x) = Cx, \quad x \in \mathbb{R}$
Apply IC, $y(0) = 5$. → Not possible to satisfy. No choice of C works.

Example Consider the IVP $y' = 2x\sqrt{y}, \quad y(0) = 0$. Can verify (see review assignment) that this has two solutions.

$$y(x) = \frac{x^4}{2}, x \in \mathbb{R}$$
$$y(x) = 0, x \in \mathbb{R}$$

General solution for DE

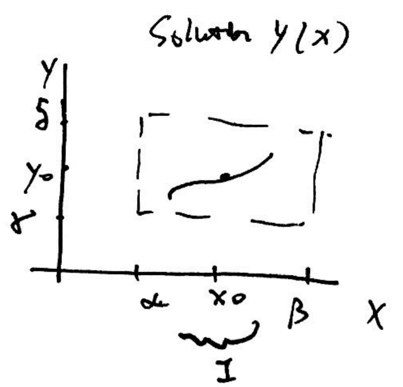
$$y(x) = (\frac{x^2}{2} + C)^2$$

(interval of existence depends on C)

How to predict if an IVP has a solution?

<u>**Theorem**</u> (Existence and Uniqueness Theorem)
Consider the IVP $\quad \dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \quad (1)$
Suppose the functions f and $\frac{\partial f}{\partial y}$ are continuous on some rectangle R in the x-y plane containing the point (x_0, y_0), then for some open interval I containing x_0 the IVP(1) has one and only one solution defined on I.

<u>picture</u>



$R = \{(x, y) \mid \alpha < x < \beta;\ \gamma < y < \delta\}$

Apply this to our examples.

1. $f(x, y) = \cos(x)$

   $\frac{\partial f}{\partial y} = 0$

   —— Both are continuous for all $(x, y) \in \mathbb{R}^2$

   The Existence and Uniqueness Theorem $\implies$ can find a solution to $\dfrac{dy}{dx} = \cos(x), \quad y(x_0) = y_0$ for any (x_0, y_0)

2. $x\dfrac{dy}{dx} = y \quad \implies$ can't apply the Theorem in this form. Rewrite this in the standard/normal form

$$\frac{dy}{dx} = \frac{y}{x}$$

$\rightarrow$ only defined on $D = \{(x, y) \mid x > 0\}$ or $D = \{(x, y) \mid x < 0\}$
Initial condition $y(0) = 5$. Since$f$ is not defined when $x = 0$, Theorem doesn't apply. No conclusion.

3. $f(x, y) = 2x\sqrt{y}$
$\frac{\partial f}{\partial y} = 2x\dfrac{1}{2\sqrt{y}} = \dfrac{x}{\sqrt{y}}$
— f, $\frac{\partial f}{\partial y}$ are continuous on $D = \{(x, y) \mid y > 0\}$
Initial condition $y(0) = 0$ (problem)
Theorem doesn't apply. No conclusion.

DEs in examples 2 and 3 both have the solution $y = 0, x \in \mathbb{R}$

Such constant solutions are important in applications.

Definition 2.1.3. An equilibrium solution of an ODE is a solution where two dependent variable is constant that is $y(x) = K$, $x \in \mathbb{R}$ for some specific $K \in \mathbb{R}$
For the first order ODE $\dfrac{dy}{dx} = f(x, y) \quad y = k$ is an equilibrium solution if $f(x, k) = 0 \quad \forall x \in \mathbb{R}$

2.2 Slope Fields / Direction Fields

Can interpret $\dfrac{dy}{dx} = f(x, y)$ geometrically.
For any given point (x, y) (where $f(x, y)$ is defined), the DE tells us the slope of the solution passing through (x, y) is $f(x, y)$. Can use a computer to do this for many points and get an idea of what the solution looks like qualitatively.

2.3 Solving First Order ODEs - Exactness

Recall the quasilinear form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad (1)$$

Suppose there is a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$
Then (1) can be written

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad (2)$$

From implicit differentiation, this is equivalent to

$$\frac{d}{dx}[F(x, y(x))] = 0$$

This can be solved using FTC.

$$F(x, y(x)) = C$$

$\rightarrow$ defines $y(x)$ implicitly as a function of x

From last class, $\quad M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad (1)$

If $\exists F(x,y)$, s.t. $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$ then

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad (2)$

$\implies \frac{d}{dx}[F(x,y(x))] = 0$

$\implies F(x,y(x)) = C \quad (3) \to$ Defines $y(x)$ implicitly

Definition 2.3.1. Any equation (1) which can be written in the from (2) for some function $F(x,y)$ is called an exact differential equation.

Example $\underbrace{\cos(x)y^3}_{M(x,y)} + \underbrace{3\sin(x)y^2}_{N(x,y)}\frac{dy}{dx} = 0$

This equation is exact $\quad F(x,y) = \sin(x)y^3 \quad$ $\frac{\partial F}{\partial x} = \cos(x)y^3 = M$, $\quad \frac{\partial F}{\partial y} = 3\sin(x)y^3 = N$

DE is equivalent to $\frac{d}{dx}[\sin(x)y^3(x)] = 0$

$$\implies \sin(x)y^3(x) = C \qquad \implies \text{defines solutions implicitly}$$

This can be verified by the following Theorem.

Theorem (Criteria for Exactness)
Suppose that the functions $M(x,y)$ and $N(x,y)$ are continuous and have continuous first order partial derivatives in the open rectangle R: $\{(x,y)|\ a < x < b,\ c < y < d\}$. Then the DE (1) is exact in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (4) at each point of R.

Idea of Proof Suppose (1) is exact $\implies \exists F$ s. t. $\quad M = \frac{\partial F}{\partial x}, \quad N = \frac{\partial F}{\partial y}$, $\quad \frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}$

By hypothesis, $\frac{\partial^2 F}{\partial y \partial x}, \frac{\partial^2 F}{\partial x \partial y}$ are continuous on R $\implies \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} \implies$ (4) holds
Suppose that (4) holds. Can use (4) to construct F. (see text chapter (6))

Examples Determine if the following DEs are exact.

Test $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies$ exact

1. $\underbrace{\cos(x)y}_{M} + \underbrace{3\sin(x)}_{N}\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = \cos(x) \neq \frac{\partial N}{\partial x} = 3\cos(x) \qquad \implies$ Not exact

2. $\underbrace{4x^3y}_{M} + \underbrace{x^4}_{N}\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x^3 \qquad \implies$ Exact

   By inspection, $F(x,y) = x^4y$
   DE: $\frac{d}{dx}[x^4y] = 0$

3. $\underbrace{x^2}_{M} + \underbrace{y^2}_{N}\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0 \qquad \implies$ Exact

Last example illustrates a whole set of exact DEs. Any DE of the form.

$M(x) + N(y)\frac{dy}{dx} = 0$ is exact where M, N and first partials are continuous

Such DE's are called separable. In normal/standard form they can be written

$$\frac{dy}{dx} = -\frac{M(x)}{N(y)} = g(x)h(y) = f(x,y)$$

where $N(y) \neq 0$

2.4 Solving Separable Equations

Example 1 $x\frac{dy}{dx} = y$ Find a general solution in explicit from.

Solution Rewrite DE $\frac{dy}{dx} = \frac{y}{x}$ $(x \neq 0)$ DE is separable.
Separate: $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$ $(x, y \neq 0)$
Integrate:
$\int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{x}dx$
$\int \frac{1}{y}dy = \int \frac{1}{x}dx \implies \ln|y| = \ln|x| + C$

Take exponential of both sides:

$e^{\ln|y|} = e^{\ln|x|+C}$
$|y| = e^C|x|$

Depending on signs of x and y there are two cases:

$y = e^C x$ or $y = -e^C x$

Collect into one formula: $y = \bar{c}x$ where $\bar{c} = \pm e^C$
This satisfies the DE for all $x \in \mathbb{R}$
Check for equilibrium (constant) solutions: $y = 0$ $x \in \mathbb{R}$ (This is included in the general formula if we allow $C = 0$)

Example Find a general solution of

$$\frac{dy}{dx} = 2x\sqrt{y}$$

Solution DE is separable.
Separate: $\frac{1}{2\sqrt{y}}\frac{dy}{dx} = x$ $(y \neq 0)$
Integrate

$$\int \frac{1}{2\sqrt{y}}dy = \int xdx$$

$$\sqrt{y} = \frac{x^2}{2} + C$$

RHS should be ≥ 0 since LHS is.
If $C < 0$, only defined for $x > \sqrt{-2C}$ or $x < -\sqrt{-2C}$
Simlify

$$y = (\frac{x^2}{2} + C)^2$$

$$C \geq 0, x \in \mathbb{R}$$

$$C < 0, x > \sqrt{-2C} \text{ or } x < -\sqrt{-2C}$$

Check for equilibnom solutions: Constants k such that $2 \times \sqrt{k} = 0, \forall x \implies k = 0$
Also have the solution $y = 0, x \in \mathbb{R}$

Exercise: Verify that when $C < 0$ the function $y = (\frac{x^2}{2} + C)^2$ only satisfies the DE if $x > \sqrt{-2C}$ or $x < -\sqrt{-2C}$

Example A motor boat is travelling in a straight line with speed 10m/s when the motor is turned off. The water provides a resistive force which is proportional to the square of the velocity and acts in the opposite direction. The mass of the boat and passengers is 1000kg. If after 1 second the speed is 9m/s, when will the speed be 2m/s?

Solution Physical law - Newton's second law
Variables:

t - time (s)

v -velocity (m/s)

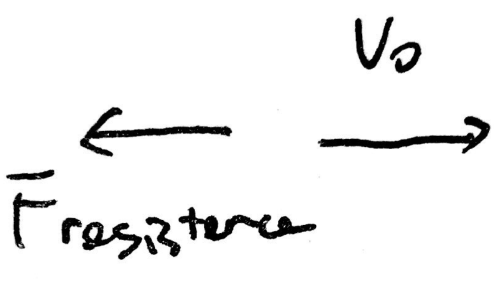
Constants: mass -1000 kg
Assumptions

- positive direction is direction of initial velocity
- $t = 0$ when the motor is turned off

Model: Newton's 2^{nd} Law:

$$m\frac{dv}{dt} = F_{\text{resistance}}$$

$$1000\frac{dv}{dt} = -kv^2$$



Note:

- minus means force acts in opposite direction of velocity
- $k > 0$ proportionally constant

Initial condition: $v(0) = 10$
Other condition: $v(1) = 9$
Solve the DE:

$$-\int v^2 dv = \int \frac{k}{1000} dt$$

$$\frac{1}{v} = \frac{kt}{1000} + C$$

Apply initial condition: $v(0) = 10, \frac{1}{10} = C$
Implicit solution of IVP $\frac{1}{v} = \frac{kt}{1000} + \frac{1}{10}$
Use $v(1) = 9:$ $\frac{1}{9} = \frac{kt}{1000} + \frac{1}{10} \implies \frac{k}{1000} + \frac{1}{90}$
Put into solution $\frac{1}{v} = \frac{t}{90} + \frac{1}{10}$

Solve for v : $v(t) = \frac{90}{t+9}$, $\underbrace{t \geq 0}_{\text{what makes sense for physical problem}}$

Let t^* be the time when $v = 2$m/s

$$v(t^*) = 2 \implies 2 = \frac{90}{t^* + 9} \implies t^* = 36$$

The speed of the boat will be 2m/s 36 seconds after the motor is turned off.

2.5 Natural growth and decay

Many natural phenomena exhibit the following property

The time rate of change of some quantity is proportional to the quantity itself

Let $y(t)$ be the quantity, t be time, a model for this is $\dfrac{dy}{dt} = \alpha\, y$ (α - constant of proportionality)

Specific Examples

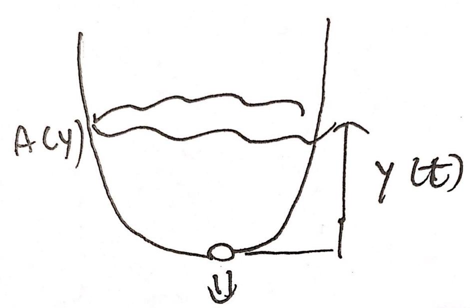
1. Let $p(t)$ be the size of a population

   t - time
   $\frac{dP}{dt} = \alpha\, P$
   α positive or negative

2. Compound interest
   $A(t)$ - amount of money in investment
   t - time (year)
   Assuming - continuously compounded interest

   $$\frac{dA}{dt} = r\, A, \quad r > 0 \quad \text{- yearly interest rate}$$

3. Application Clepsydra (water clock)
   This is a device that measures time by the flow of water into or out of a container. We will develop a model for this.
   Consider a container with a hole at the bottom through which water flows.



Let $y(t)$ be the height of the water at time t and $V(t)$ the corresponding volume. It can only be shown that the speed of the water exiting the hole at time t is $\gamma\sqrt{2gy(t)}$ where

- g is gravitational acceleration
- $\sqrt{2gy}$ comes from Fluid Dynamics
- γ accounts for friction in the hole $0 < \gamma \leq 1$

Let a be the area of the hole at bottom.
By conservation of mass:

rate of change of volume in tank = rate water leaves tank

$$\frac{dV}{dt} = -a\gamma\sqrt{2gy}$$

or

$$\frac{dV}{dt} = -k\sqrt{y} \qquad (1) \qquad \text{Toricelli's Law}$$

Let $A(h)$ be the cross sectional area of the container a height h from the bottom.

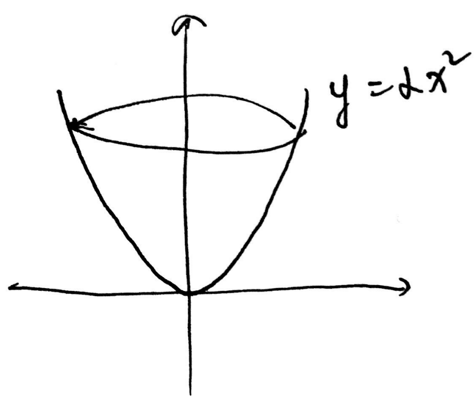
Then the volume of water is $V(y) = \int_0^y A(u)du$

$$\implies \frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}$$

So model (1) can be written

$$A(y)\frac{dy}{dt} = -k\sqrt{y} \qquad \rightarrow \text{ Nonlinear, separate DE for } y(t)$$

Example: Consider a container where the shape corresponds to the surface of revolution of a parabola, $y = \alpha x^2 (\alpha > 0)$ rotated around the y-axis. If the initial height is y_0, what time will it be empty?



$$A(y) = \pi x^2$$
$$= \frac{\pi}{2} y$$

The model becomes

$$\frac{\pi}{2} y \frac{dy}{dt} = -k\sqrt{y}, \qquad y(0) = y_0$$

$$y\frac{dy}{dt} = -\bar{k}\sqrt{y}, \qquad y(0) = y_0$$

Equilibrium solution $y(t) = 0, t \in \mathbb{R}$
Solve the DE

$$\sqrt{y}\frac{dy}{dt} = -\bar{k}$$

$$\int y^{1/2} dy = -\int \bar{k} dt$$

$$\frac{2}{3} y^{3/2} = -\bar{k}t + C$$

Apply IC: $y(0) = y_0 \implies \frac{2}{3}y_0^{3/2} = C$

Solution of IVP: $\frac{2}{3}y^{3/2} = -\bar{k}t + \frac{2}{3}y_0^{3/2}$ (implicit form)

$$y(t) = (y_0^{3/2} - \frac{2}{3}\bar{k}t)^{2/3}$$

Container will be empty when $y(t) = 0$

$$t = \frac{2}{3}\frac{y_0^{3/2}}{\bar{k}}$$

2.6 Integrating Factors

Recall some examples:

$\cos(x)y + 3\sin(x)\frac{dy}{dx} = 0 \implies$ Not exact

$\cos(x)y^3 + 3y^2\sin(x)\frac{dy}{dx} = 0 \implies$ Exact

Multiplying the first DE by y^2 makes it exact
The two DEs have the same general solution: $y(x) = [\frac{C}{\sin(x)}]^{1/3}$

Suppose the DE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ (1)

is not exact but $v(x,y)M(x,y) + v(x,y)N(x,y)\frac{dy}{dx} = 0$ (2)

is exact. Then $v(x,y)$ is called an integrating factor for (1)

Note: Every solution of (1) is a solution of (2) but (2) may have solutions which are not solutions of (1). These are given by $v(x, y(x)) = 0$
There is no general method for finding $v(x,y)$.
Except in some special cases

(1) separable DEs

(2) DEs in the form: $\underbrace{P(x)y}_{M} + \underbrace{1}_{N}\frac{dy}{dx} = 0$

$$P(x) \not\equiv 0$$

$\frac{\partial M}{\partial y} = P(x), \quad \frac{\partial N}{\partial x} = 0 \implies$ Not exact

Suppose $v(x)$ is an integrating factor.
Then

$$\underbrace{v(x)P(x)y}_{\bar{M}} + \underbrace{v(x)}_{\bar{N}}\frac{dy}{dx} = 0 \quad \text{is exact}$$

$\implies$

$$\frac{\partial \bar{M}}{\partial y} = \frac{\partial \bar{N}}{\partial x}$$

$v(x)P(x) = \frac{dv}{dx} \implies$ A separable DE for $v(x)$

$$\int \frac{dv}{v} = \int P(x)dx$$

$\bullet\bullet\bullet$ $v(x) = Ke^{\int P(x)dx}$, K an arbitrary constant
usually we take $K = 1$

From last lecture $P(x)y + \dfrac{dy}{dx} = 0$ has integrating factor $e^{\int P(x)dx}$
We can find the solution of the DE as

$$e^{\int P(x)dx}P(x)y + e^{\int P(x)dx}P(x)\frac{dy}{dx} = 0$$
$$\frac{d}{dx}[e^{\int P(x)dx}y(x)] = 0$$

Implicit Differentiation

General Solution $e^{\int P(x)dx}P(x)y(x) = C$, C an arbitrary constant.

Recall our general first order ODE $H(x, y, \frac{dy}{dx}) = 0$

Definition A linear first order ODE is one where H is a linear function of y and $\frac{dy}{dx}$
Linear DE's can always can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (1)$$

Example Find the solution of $xy' + 2y = 4x^2$, $y(-1) = 2$

Solution Rewrite the DE in standard form

$$y' + \frac{2}{x}y = 4x, \quad x \neq 0$$

$P(x) = \dfrac{2}{x}$, Integrating factor

$$\begin{aligned} v(x) &= e^{\int \frac{2}{x}dx} \\ &= e^{2\ln|x|} \\ &= e^{\ln(x^2)} = x^2 \end{aligned}$$

Multiply through the DE by $v(x) = x^2$

$$x^2y' + 2xy = 4x^3$$
$$\frac{d}{dx}[x^2y(x)] = 4x^3$$

Product Rule + Implicit Differentiation

$$\int \frac{d}{dx}[x^2y(x)]dx = \int 4x^3dx$$
$$x^2y(x) = x^4 + C$$

General Solution: $y(x) = x^2 + \dfrac{C}{x^2}$
Apply initial condition: $y(-1) = 2$

$$2 = 1 + \frac{C}{1} \implies C = 1$$

IVP has a unique solution: $y(x) = x^2 + \dfrac{1}{x^2}$, $x < 0$

Theorem (Existence and Uniqueness for Linear DEs)
Suppose the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing x_0. Then, for any y_0 the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0$$

Has a unique solution defined on I.

Proof Since $P(x), Q(x)$ are continuous on I, so are

$$\int P(x)dx, \quad v(x) = e^{\int P(x)dx}, \quad \int v(x)Q(x)dx$$

And $v(x) \neq 0$ on I
We showed general solution of the DE is

$$y(x) = \frac{C}{v(x)} + \frac{1}{v(x)} \int v(x)Q(x)dx$$

This is continuous on I, and satisfies DE.
For any $x_0 \in I$ rewrite this as

$$y(x) = \frac{\bar{C}}{v(x)} + \frac{1}{v(x)} \int_{x_0}^{x} v(s)Q(s)ds$$

Apply the IC: $y(x_0) = y_0$

$$y_0 = \frac{\bar{C}}{v(x_0)} \implies \bar{C} = y_0 v(x_0)$$

Thus there is a unique value of C that satisfies the IC and hence a unique solution to the IVP. □

Applications Newton's Law of Cooling/Heating
The rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the ambient temperature (surroundings).

Model

T: temperature of object

t: time

$A(t)$: ambient temperature

$$\frac{dT}{dt} = K(A(t) - T) \qquad K > 0 \text{ constant}$$

2.6.1 Tutorial 2

To get 100% on assignment

1. Answer all questions.
2. Theory reasoning must be based on exact Definitions and Theorems (Title # 1 page source)

Thm 1 (Uniqueness + Existence) p.26 + addit.

chapter 1.4

 # 12 ODE $yy' = x(y^2+1)$

$$y\frac{dy}{dx} = x(y^2+1)$$

$$\int \frac{y}{y^2+1} dy = \int x\ dx$$

Substitute $u = y^2+1, \qquad du = 2y\ dy$

$$\implies \frac{1}{2}\int \frac{du}{u} = \int x\ dx$$

$$\ln|u| = x^2 + C$$

$$e^{\ln|u|} = e^{x^2+C}$$

$$u = e^{x^2}\cdot e^C$$

$$y^2+1 = Ce^{x^2}$$

Note that C is not the same in different equations...

$$y = \pm\sqrt{Ce^{x^2}-1}$$

However, we need $Ce^{x^2} - 1 \geq 0$, so we must have restrictions on C. To make it easier to write, we have

$$y = \pm\sqrt{e^{x^2+C}-1}, \qquad C \geq 0$$

#21 $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}$

Immediately, we get $x \in (-\infty,-4)\cup(4,+\infty)$ Since we need $\exists f(x)$
This is IVP: ODE and IC $(y(5)=2)$
Solving it we get: $y^2 = \sqrt{x^2-16}+C$
Apply IC, we get: $C = 1$

$$y = \sqrt{\sqrt{x^2-16}+1}, \quad x > 4$$

 # 30 $(\frac{dy}{dx})^2 = 4y$

Taking square roots, $\frac{dy}{dx} = \pm 2y^{\frac{1}{2}}$
$y(x) = (x \pm C)^2,$ and we lose $y(x) \equiv 0$

Given (a,b)

(a) solution doesn't exist

(b) ∞ solutions (we can also have piecewise solution, but should be continuous)

(c) finite solutions

2.7 Application Cont'd

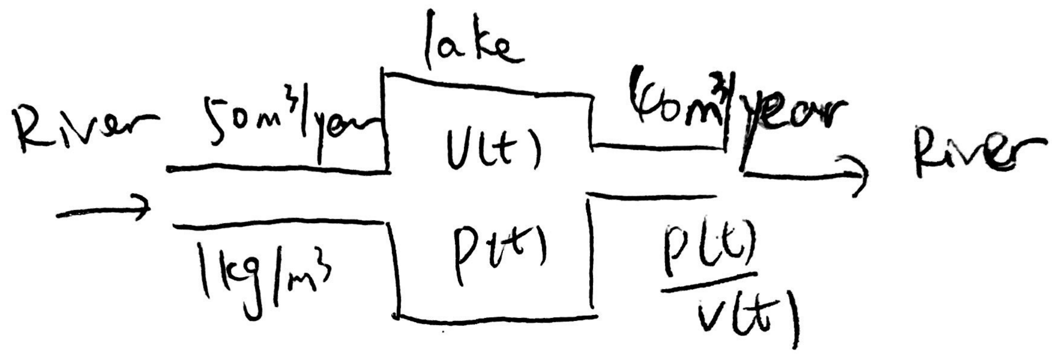
Application Mixture / compartment problems

Example A lake contains 10^4 m^3 of clean water when a factory starts dumping waste into a river that feeds the lake at 50 m^3/year. The concentration of pollution in the river is 1kg/m^3. Another river drains water from the lake at a rate 40 m^3/year.
Find the amount of pollution in the lake at any time after the dumping starts.

Model

- V: volume of the lake (m^3)
- p: mass of pollution in the lake (kg)
- t: time (years)

Physical principle - conservation of mass



Assume Lake is "well-mixed", so $p(t)$ same everywhere

Rate of change of V in time: $\begin{matrix} \textit{rate} \\ \textit{water} \\ \textit{in} \end{matrix} - \begin{matrix} \textit{rate} \\ \textit{water} \\ \textit{out} \end{matrix}$

$$\frac{dV}{dt} = 50 - 40, \qquad v(0) = 10^4 \tag{1}$$

$\Longrightarrow$ Easily solved: $v(t) = 10t + 10^4$

Rate of change of p in time = $\begin{matrix} \textit{rate} \\ \textit{pollution} \\ \textit{in} \end{matrix} - \begin{matrix} \textit{rate} \\ \textit{pollution} \\ \textit{out} \end{matrix}$

= (rate water in) (concentration pollutions in) − (rate water out) (concentration pollution out)

$$\frac{dp}{dt} = 50 \cdot 1 - 40 \cdot \frac{p(t)}{V(t)}$$

Use the expression for $V(t)$, and rearrange

$$\frac{dp}{dt} + \frac{40}{10t + 10^4} p = 50 \qquad \text{(Linear DE for } p(t))$$

Initial condition: $p(0) = 0$ (no pollution at start)

Integrating factor: $v(t) = e^{\int \frac{40}{10t+10^4} dt} = e^{\int \frac{4}{t+10^3} dt} = \ldots = (t + 10^3)^4$

Use $v(t)$ to solve DE:

$$(t + 10^3)^4 \frac{dp}{dt} + 4(t + 10^3)^3 p = 50(t + 10^3)^4$$

$$\frac{d}{dt}[(t + 10^3)^4 p] = 50(t + 10^3)^4$$

Integrate wrt t

$$(t + 10^3)^4 p = 10(t + 10^3)^5 + C$$

Apply IC, $p(0) = 0 \implies \cdots C = -10^{16}$
Put this into solution, and simplify, we get
Amount of pollution at time t:

$$p(t) = 10(t + 10^3) - \frac{10^{16}}{(t + 10^3)^4}$$

2.8 Solving DEs using a change of variables

Definition A homogeneous first order ODE is one that can be written in the form $\frac{dy}{dx} = F(\frac{y}{x})$ (1)
This kind of DE can be solved using the following change of variables.

$$\text{Let } v(x) = \frac{y(x)}{x}. \text{ Then } y = xv, \qquad \text{and } \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Putting this into (1)

$$v + x\frac{dv}{dx} = F(v)$$

$$\frac{dv}{dx} = \frac{F(v) - v}{x} \qquad \leftarrow \text{A separable DE}$$

Example $\dfrac{dy}{dx} = 2\dfrac{x}{y} + \dfrac{y}{x}$

Use the change of variables above: $y = xv$

$$v + x\frac{dv}{dx} = \frac{2}{v} + v$$

$$x\frac{dv}{dx} = \frac{2}{v}$$

Solve as a separable DE

$$\int v\ dv = \int \frac{2}{x} dx$$

$$\frac{v^2}{2} = 2\ln|x| + C = \ln(x^2) + C$$

Put $v(x) = \frac{y(x)}{x}$

$$\frac{1}{2} \cdot \frac{y^2}{x^2} = \ln(x^2) + C \implies y^2 = 2x^2\ln(x^2) + 2Cx^2$$

Definition A Bernoulli Equation is a first order ODE that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

If $n = 0$ or $n = 1$, then DE is linear.
If $n \neq 0, 1$ use the change of variables: $v(x) = [y(x)]^{1-n}$

$$\begin{aligned}\frac{dv}{dx} &= (1-n)y^{-n}\frac{dy}{dx} \\ &= (1-n)y^{-n}[-P(x)y + Q(x)y^n] \\ &= -(1-n)y^{1-n}P(x) + (1-n)Q(x) \\ &= -(1-n)P(x)v + (1-n)Q(x)\end{aligned}$$

This is a linear DE for $v(x)$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Example $\frac{dy}{dx} = 2\frac{x}{y} + \frac{y}{x}$
$\frac{dy}{dx} - \frac{1}{x}y = 2xy^{-1}$

Bernoulli DE with $n = -1$
Change of variables:

$$v = y^{1-(-1)} = y^2$$

$$\frac{dv}{dx} = 2y\frac{dy}{dx}$$

$$\begin{aligned}\frac{dv}{dx} &= 2y[\frac{1}{x} \cdot y + 2xy^{-1}] \\ &= \frac{2}{x}y^2 + 4x\end{aligned}$$

Then,

$$\frac{dv}{dx} - \frac{2}{x}v = 4x$$

Exercise: Solve and show you get the same answer as before.

Solution: We can find that integrating factor is $u(x) = \frac{1}{x^2}$
Then we get

$$\frac{d}{dx}[\frac{1}{x^2}v(x)] = \frac{4}{x}$$

$$v(x) = 4\ln(x)x^2 + Cx^2$$

Since $v(x) = y^2$, thus

$$y^2 = 4x^2\ln(x) + Cx^2$$

2.9 Second order ODEs reducible to first order

Most general second order ODE:

$$H(x, y, y', y'') = 0 \qquad (1)$$

If the dependent variable is missing in (1) we have

$$H(x, y', y'') = 0 \qquad (2)$$

Can solve as follow:

$$\begin{aligned}\text{Let } y' &= v \qquad (3)\\ \text{then } y'' &= v' \qquad (4)\end{aligned}$$

Putting these into (2) gives

$$H(x, v, v') = 0 \qquad (5) \rightarrow \text{ first order ODE for} v$$

Can solve (5) for v and (3) for y

Example $xy'' = y'$

Note: $'$ means $\frac{d}{dx}$

Let $v = y', \quad v' = y''$

$$xv' = v$$
$$x\frac{dv}{dx} = v$$

We have solved this before: $v(x) = C_1 x$
Now solve for y

$$y' = v$$
$$\frac{dy}{dx} = C_1 x$$

Just integrate wrt x $\quad y(x) = C_1\frac{x^2}{2} + C_2, \quad x \in \mathbb{R}$ (two arbitrary constants)

If the independent variable is missing from (1) we have

$$H(y, y', y'') = 0 \qquad (6)$$

This may be solve via the change of varibles

$$y' = v(y) \qquad (7)$$
$$\frac{dy}{dx} = v(y(x))$$

Then

$$y'' = \frac{d^2y}{dx^2} = \frac{dv}{dy}\cdot\frac{dy}{dx} = \frac{dv}{dy}v \qquad (8)$$

Putting these into (6) gives

$$H(y, v, v\frac{dv}{dy}) = 0 \qquad (9)$$

First order ODE for $v(y)$

Solve (9) for $v(y)$ and (7) for $y(x)$

Example $yy'' = (y')^2$
Let $v(y) = y', \quad v\frac{dv}{dy} = y''$

$$yv\frac{dv}{dy} = v^2$$

$\rightarrow$ separable DE for $v(y)$

Note this has solution $v = 0$,
For $v \neq 0$ $\quad y\frac{dv}{dy} = v$

Solve this before: $v(y) = C_1 y$

...(exercise) $\implies y(x) = C_2 e^{C_1 x}$

3

Mathematical Models + Numerical Methods

3.1 Population Models

Let $P(t)$ be the number of individuals in a population as a function of time (t).

Basis model: $\dfrac{dP}{dt} = \text{birth rate} - \text{death rate}$

Simplest assumption: birth rate and death rates are proportional to the population size.

$$\frac{dP}{dt} = \beta P - \delta P \qquad \beta,\ \delta > 0$$

$$\frac{dP}{dt} = (\beta - \delta)P \qquad \to \text{exponential growth or decay depending on whether } \beta > \delta \text{ or } \beta < \delta$$

More complicated model birth rate decreases with population size due to lack of resources

$$\frac{dP}{dt} = (\underbrace{\beta_0 - \beta_1 P}_{\beta} - \delta)P \qquad \beta_0,\ \beta_1,\ \delta > 0$$

Rewrite:

$$\frac{dP}{dt} = (\beta_0 - \delta)P - \beta_1 P^2$$

If $(\beta_0 - \delta) > 0$ this is called the Logistic Model and is usually written

$$\frac{dP}{dt} = kP(M - P) \qquad k,\ M > 0$$

Note:

(1) if you apply the Existence and Uniqueness Theorem to this DE $\to$ guarantee a unique solution for any IC $P(t_0) = P_0$

(2) The DE has two equilibrium solutions

- $P(t) = 0, \quad t \in \mathbb{R}$
- $P(t) = M, \quad t \in \mathbb{R}$

Solve the DE as a separable DE (it is also a Bernoulli DE)

$$\int \frac{dP}{P(M-P)} = \int k\ dt \qquad (P \neq 0,\ P \neq M)$$

Using Partial Fractions:

$$\frac{P_{(t)}}{M - P_{(t)}} = Ce^{kMt}$$

(C arbitrary constant)

Initial conditions: $P(0) = P_0$ $\implies C = \frac{P_0}{M-P_0}$
Putting this into the solution and rearranging (exercise)

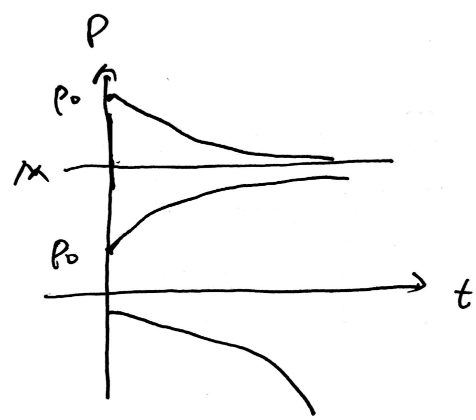
$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

$$\implies \begin{array}{ccc} P_0 = 0 & \rightarrow & P(t) = 0, t \in \mathbb{R} \\ P_0 = M & \rightarrow & P(t) = M, t \in \mathbb{R} \end{array}$$

Limiting behaviours:

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}} = M$$

if $P_0 > 0$



$$\begin{cases} \frac{dP}{dt} > 0 & \text{if } 0 < P < M \\ \frac{dP}{dt} < 0 & \text{if } P > M \\ \frac{dP}{dt} < 0 & \text{if } P < 0 \end{cases}$$

Interpretation If $P_0 > 0$, then the population tends toward M as $t \to \infty$.

M is called the carrying capacity of the population

3.1.1 Equilibrium solutions and stability

Focus on DE's of the form $y' = f(y)$ (autonomous DEs)

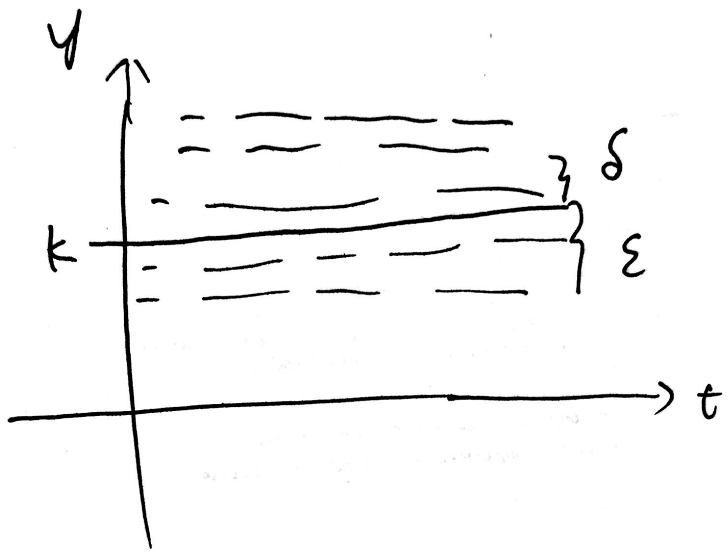
Equilibrium points: $y(t) = K, t \in \mathbb{R}$ $f(k) = 0$

Assume eg(1) with IC $y(0) = y_0$ has a unique solution.

Definition An equilibrium solution of (1) is stable if for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|y_0 - k| < \delta \implies |y(t) - k| < \epsilon \quad \forall t \geq 0$$

otherwise it is unstable.



An equilibrium solution of (1) is asymptotically stable if it is stable and there is $\eta > 0$ such that

$$|y_0 - k| < \eta \implies \lim_{t\to\infty} y(t) = k$$

In logistic models:

- $P(t) = M$ is asymptotically stable
- $P(t) = 0$ is unstable

3.1.2 Logistic Growth with Harvesting

Example (Logistic Growth with Harvesting)
Consider a population of fish in a lake that obeys the logistic model. Suppose we wish to allow fishing at a constant rate h. How should we choose h so that the population doesn't go extinct?

Model

- $P(t)$ - population at time t
- k, M - as in logistic model
- $t = 0$ - when fishing start
- P_0 - population when fishing starts

$$\frac{dP}{dt} = \underbrace{kP(M-P)}_{\text{logistic growth}} \quad \underbrace{- \quad h}_{\text{loss due to fishing}}$$

Equilibrium solutions

$$kP(M-P)-h=0$$

$$-k[P^2-MP+\frac{h}{k}]=0$$

$$P=\frac{M\pm\sqrt{M^2-4\frac{h}{k}}}{2}=M_{\pm}$$

Three cases:

- $M^2-4h/k>0$: two equilibrium solutions: $0<M_-<M_+<M$
- $M^2-4h/k=0$: one equilibrium solution: $P(t)=\frac{M}{2}$
- $M^2-4h/k<0$: no equilibrium solution

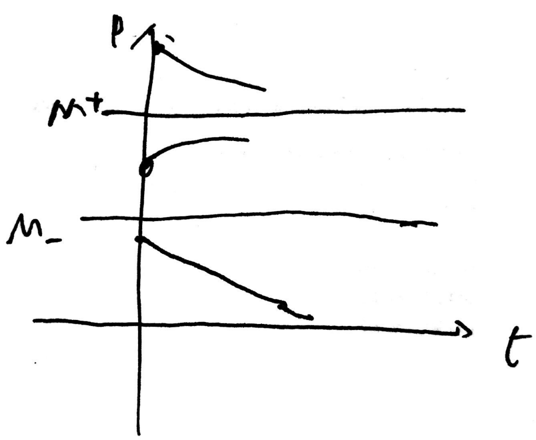
Case (1) Rewrite the model

$$\frac{dP}{dt}=k(P-M_-)(M_+-P),\quad P(0)=P_0$$

From the DE we can see:

$$\begin{cases}\frac{dP}{dt}<0 & \text{if } P<M_- \\ \frac{dP}{dt}>0 & \text{if } M_-<P<M_+ \\ \frac{dP}{dt}<0 & \text{if } P>M_+\end{cases}$$

Can predict what solutions look like



Prediction: M_+ is (asymptotically) stable
M_- is unstable

Can show that the solution of the IVP is

$$P(t)=\frac{M_+(P_0-M_-)-M_-(P_0-M_+)e^{-k(M_+-M_-)t}}{(P_0-M_-)-(P_0-M_+)e^{-k(M_+-M_-)t}}$$

$$\lim_{t\to\infty}P(t)=M_+\qquad \text{if } P_0>M_-$$

Can show solutions look as we predicted

Interpretation

- $0 < P_0 < M_-$: population $\to 0$ in finite time
- $M_- < P_0 < M_+$: population increases to M_+
- $M_+ < P_0$: population decreases to M_+

To ensure population doesn't go extinct, choose h so that

- $M^2 > \frac{h}{k}$
- $M_- \leq P_0$

3.1.3 Tutorial 3

Step 1 Standard form: $y' + P(x)y = Q(x)$

Step 2 $\rho(x) = e^{\int P(x)dx}$

Step 3 $\rho(x)y(x) = \int \rho Q + C$

In Section 1.5 DE

#7 $2xy' + y = 10\sqrt{x} \qquad D : x \geq 0$
We are going to divide both sides by $2x, (x \neq 0)$
Check $y \equiv 0$. $y(0) = 0 \implies y \equiv 0$

$$y' + \underbrace{\frac{1}{2x}}_{P(x)} y = \underbrace{\frac{5}{\sqrt{x}}}_{Q(x)}$$

Find the integrating factor, $\rho(x) = e^{\int \frac{1}{2x} dx} = \sqrt{x}$

$$\sqrt{x}\, y(x) = 5x + C$$

$$\begin{cases} y(x) = 5\sqrt{x} + \frac{C}{\sqrt{x}}, & x_0 > 0 \\ y(x) \equiv 0, & x_0 = 0 \end{cases}$$

#? $y' + 2xy = x \quad y(0) = 2$
$\rho(x) = \ldots = e^{x^2}$
$\vdots$ done with some substitution in integral
$y(x) = \frac{1}{2} - Ce^{-x^2} \overset{\text{Apply IC}}{\implies} y(x) = \frac{1}{2} - \frac{5}{2}e^{-x^2}$

Word Problem
Words $\overset{\text{translate}}{\implies}$ Math $\overset{\text{translate}}{\implies}$ Words (answer in words)

- denote all variables and parameters, units
- DE

#36

- t - time, min, ...
- $x(t)$ - amount of salt in tank at time t, 1b
- V_0 - amount of water in tank at time t

$0 \le t \le 60\,\mathrm{min}$

$$\frac{dx}{dt} = r_i c_i - r_0 c_0 \quad F(x, y)$$

$$x' + \frac{3}{60 - t} x = 2$$

$$x(t) = (60 - t) + C(60 - t)^3 \overset{x(0)=0}{\Longrightarrow} C = -\frac{1}{3600}$$

Find $x' = 0, x'' < 0$, and check endpoints

Newton's Law of Cooling

$$\frac{dT}{dt} = k(A - T) \qquad T(0) = T_0$$

$$A(t) = 25 - 20e^{0.1t}$$

$$\frac{dT}{dt} + kT = (25 - 20e^{0.1t})k$$

$$\frac{d}{dt}[e^{kt}T] = 25ke^{kt} - 20k^{(k-0.1)t}$$

(1) $k = 0.1$

$$T(t) = 25 - 20kte^{-kt}$$

(2) $k \neq 0.1$

$$T(t) = 25 - \frac{20}{k - 0.1}(e^{kt} - e^{-0.1t})$$

3.2 Position and Velocity Problems

Air resistance A force that resists the motion of an object.

- Magnitude - proportional to a power of the speed
- Direction - opposite to the direction of the motion

If v is the velocity of an object

$$F_{\mathrm{AR}} = \begin{cases} -k|v|^p & \text{if } v > 0 \\ k|v|^p & \text{if } v < 0 \end{cases}$$

where $k > 0$, $1 \le p \le 2$. k, p depends on the object.

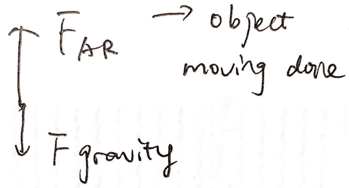
$$F_{\mathrm{AR}} = -kv|v|^{p-1}$$

$p = 1 \quad F_{\mathrm{AR}} = -kv$

$p = 2 \quad F_{\mathrm{AR}} = -kv|v|$

Example An object is thrown downward form height y_0 with speed v_0. The object encounters air resistance proportional to the square of the speed. Find the velocity of the object.

Solution Diagram of forces



Assumptions

- F_{gravity} is constant
- Up is the positive direction

Model

- v - velocity
- t - time
- m - mass of object
- g - gravitational acceleration
- $k > 0$ - coefficient of A.R

Applying Newton's 2nd Law $m\frac{dv}{dt} = F_{\text{gravity}} + F_{\text{AR}}$

$$m\frac{dv}{dt} = -mg - kv|v|$$

$$v(0) = -v_0$$

Since object is falling, $v < 0 \implies -v|v| = v^2$

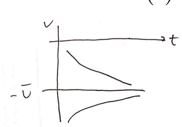
$$m\frac{dv}{dt} = -mg + kv^2$$

Equilibrium solutions: $-g + \dfrac{k}{m}v^2 = 0 \implies v = \pm\sqrt{\dfrac{mg}{k}} = \pm\overline{v}$

Rewrite the DE: $\frac{dv}{dt} = \frac{k}{m}(v^2 - \overline{v}^2) = \frac{k}{m}(v - \overline{v})(v + \overline{v})$

Relevant equilibrium solution is $v(t) = -\overline{v}, t \in \mathbb{R}$

$$\begin{cases} \frac{dv}{dt} > 0 & \text{if } v < -\overline{v} \\ \frac{dv}{dt} < 0 & \text{if } v > -\overline{v} \end{cases}$$



Predict: $v(t) = -\overline{v}$ is asymptotically stable

Solving the DE using partial fractions

$$v(t) = \overline{v} \cdot \frac{\left[1 + Ce^{\frac{2k}{m}\overline{v}t}\right]}{\left[1 - Ce^{\frac{2k}{m}\overline{v}t}\right]}$$

where C is arbitrary

Apply IC: $v(0) = -v_0 \implies C = \frac{v_0 + \overline{v}}{v_0 - \overline{v}}$

Solution for velocity:

$$v(t) = \overline{v} \cdot \frac{\left[v_0 - \overline{v} + (v_0 + \overline{v})e^{\frac{2k}{m}\overline{v}t}\right]}{\left[v_0 - \overline{v} - (v_0 + \overline{v})e^{\frac{2k}{m}\overline{v}t}\right]}$$

$$\lim_{t\to\infty} = -\overline{v} = -\sqrt{\frac{mg}{k}}$$

After a long time the velocity is effectively constant the value $-\sqrt{\frac{mg}{k}}$ is called the terminal velocity

3.2.1 Newton's Law of Gravitation

In previous examples, we used $F_{\text{gravity}} = mg$. This is an approximation of a more general law.

Two objects of masses m_1 and m_2, separated by distance r. For motion in one dimension:

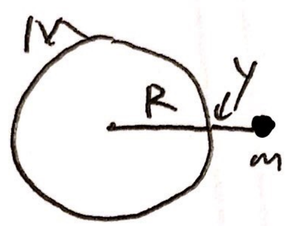
$$|F_{\text{gravity}}| = G\frac{m_1 m_2}{r^2}$$

where $G = 6.674 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$

For an object of mass m, a distance y above the surface of the earth

$$|F_{\text{gravity}}| = G\frac{m_1 m_2}{(R+y)^2}$$

where M = mass of earth = 5.972×10^{24}kg $\quad R$ = radius of earth = 6.371×10^6m



If y is small

$$G\frac{Mm}{(R+y)^2} \approx G\frac{Mm}{R^2} = m\underbrace{\frac{Gm}{R^2}}_{g}$$

Example A projectile is fixed vertically upward with velocity v_0 from the surface of the earth. Assuming gravity is the only force acting, but taking into account the variation of this force with distance, find the velocity.

Solution Diagram of forces $\downarrow F_{\text{gravity}}$

Assume up is positive direction

- v - velocity of object
- t - time
- y - position of object above surface of earth
- v_0 - initial velocity
- m - mass of object
- M - mass of earth
- R - radius of earth
- G - universal gravitational constant

Apply Newton's 2$^{\text{nd}}$ Law: $m\frac{dv}{dt} = F_{\text{gravity}}$

$$m\frac{dv}{dt} = -\frac{GMm}{(R+y)^2}$$

Initial conditions: $v(0) = v_0 \qquad y(0) = 0$
DE: $\frac{dv}{dt} = -\frac{GM}{(R+y)^2}$
Let $r(t) = R + y(t), \qquad \frac{dv}{dt} = \frac{d^2y}{dt^2} = \frac{d^2r}{dt^2}$

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

RHS doesn't depend on t: Think of v as a function of r
Let $v = \frac{dr}{dt}, \qquad \frac{d^2r}{dt^2} = \frac{dv}{dr}\cdot\frac{dr}{dt} = v\frac{dv}{dr}$

$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$

Separable DE: $\frac{1}{2}v^2 = \frac{GM}{r} + C$
Initial condition: $t = 0 : v = v_0, y = 0 \quad \implies r = R \qquad v_{(r=R)} = v_0$

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C \implies C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2(r) = v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R})$$

We take positive sign as object moving upward

$$v(r) = \sqrt{v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R})}$$

What is the interval of existence? Need $v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R}) > 0$
Two possibilities:

(1) $v_0^2 - \frac{2GM}{R} < 0$

$r < \frac{R}{\left(1 - \frac{Rv_0^2}{2GM}\right)} = r_{\max} \implies v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R}) > 0$
Mathematically, the solution is defined for $0 < r < r_{\max}$
Physically, the solution is defined for $R \le r < r_{\max}$

(2) $v_0^2 - \frac{2GM}{R} \ge 0$. Then $v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R}) > 0$ for $r > 0$
Mathematically, the solution is defined for $0 < r$
Physically, the solution is defined for $R \le r$

Interpretation

(1) $v(r) \to 0$ as $r \to r_{\max}$
Velocity goes to zero at maximum height
TO continue the solution, think about using solution with $-\sqrt{\ldots}$

(2) $v(r) > 0$ no matter how big r is.
Observe "escapes" from the earth's gravity. The minimum initial velocity needed to achieve this is $v_0 = \sqrt{\frac{2GM}{R}}$ ——————— called the escape velocity

In either case, we can use the solution to define IVP for the position

$$\frac{dr}{dt} = \underbrace{v(r)}_{\text{solution found above}}, \quad r(0) = R$$

3.3 Numerical Approximation of Solutions of DEs

So far - only considered DE's where we can write the solution in terms of elementary functions. (powers, trig functions, exp)
This is not always the case.

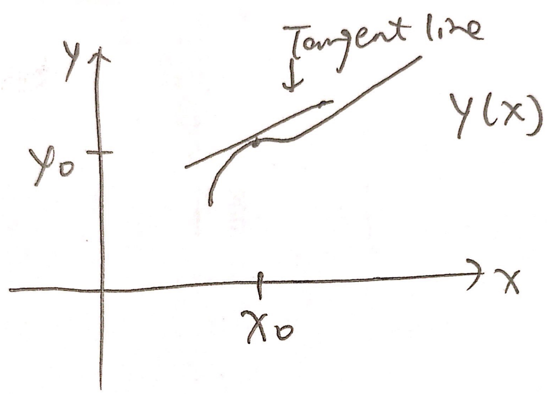
Example $\frac{dy}{dx} = e^{-x^2}$ - Linear, separable.
Has an unique solution for any IC $y(x_0) = y_0$. Can't express solution in terms of elementary function

$$y(x) = y_0 + \int_{x_0}^{x} e^{-t^2}\, dt$$

More generally, suppose we want to approximate the solution of

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0 \qquad (1)$$

Recall The tangent line (linear) approximation for $y(x)$ at the point (x_0, y_0)



$$y(x) \approx y(x_0) + y'(x_0)(x - x_0)$$

good approximation if x is close to x_0
Here $y(x)$ is unknown but $y'(x_0)$ is known from the DE.

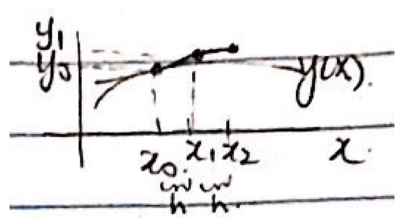
Euler's Idea Use the tangent line approximation iteratively to approximate the solution of (1) at some particular values of x

From last class: tangent line approximation to $y(x)$ at (x_0, y_0)

$$y(x) \approx y(x_0) + y'(x_0)(x - x_0)$$

Euler's Method Use this to approximate solution of

$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0$$



Look at solution at set of values for $x : x_n = x_0 + nh, \quad n = 1, 2, \ldots \quad h$ is the step size.
For $x_0 \le x \le x_1$, use the tangent line approximation at (x_0, y_0)

$$y(x) \approx y_0 + f(x_0, y_0)(x - x_0)$$

At $x = x_1$, $\quad y(x_1) \approx y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + hf(x_0, y_0) = y_1$
Use this to generate new approximation for $x_1 \leq x \leq x_2$

$$y_1(x) \approx y_1 + f(x_1, y_1)(x - x_1)$$

Summary Euler Method Algorithm

1. (Discretization) Choose step size h. Let $x_n = x_0 + nh, \quad n = 0, 1, \ldots$ (grid points)
2. (Approximation) Approximate $y(t)$ at grid points

$$y(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n, y_n)$$

3. (Interpolation) Approximate $y(x)$ between grid points

$$y(x) \approx y_n + (x - x_n)f(x_n, y_n), \quad n = 0, 1, \ldots \qquad x_n \leq x \leq x_{n+1}$$

Example $\frac{dy}{dx} = x + \frac{1}{5}y, \quad y(0) = -3$
Find approximation of solution using Euler Method for $0 \leq x \leq 5$

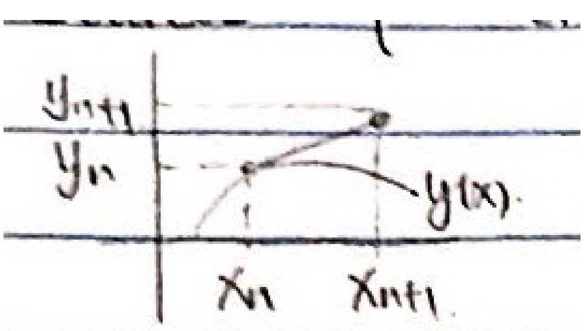
Solution Setup :

$$x_0 = 0, y_0 = -3$$
$$x_n = x_0 + nh = nh, \qquad n = 0, 1, \ldots$$
$$\begin{aligned} y_{n+1} &= y_n + hf(x_n, y_n) \\ &= y_n + h[x_n + \frac{1}{5}y_n] \end{aligned}$$

Take $h = 1, \qquad x_n = n, \quad n = 0, 1, 2, 3, 4, 5$

$$\begin{aligned} x_0 = 0, \qquad & y_0 = -3 \\ x_1 = 1, \qquad & y_1 = y_0 + [x_0 + \frac{1}{5}y_0] \\ & \quad = -3 + [0 - \frac{3}{5}] \\ & \quad = -3.6 \\ x_2 = 2, \qquad & y_2 = y_1 + [x_1 + \frac{1}{5}y_1] \\ & \quad = -3.6 + [0 - \frac{3.6}{5}] \\ & \quad = -3.32 \end{aligned}$$

3.3.1 Sources of error in Euler's Method



$$y(x_{n+1}) \approx y_n + hf(x_n, y_n)$$

(1) Error due to direct approximation

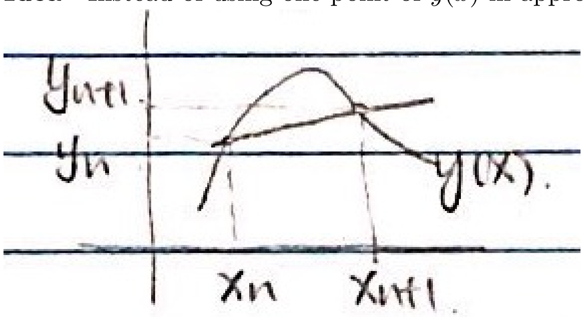
(2) Error due to the fact that $y(x_n) = y_n$

Can show (see AMATH 332) the error in solving $y' = f(x,y); y(x_n) = y_n$, for $a \leq x \leq b$ satisfies $|y_n - y(x_n)| \leq ch$ for all n.

c depend on f, and $[a,b)$

One way to reduce error: make h smaller.
Another way: improve the approximation step.

Idea Instead of using one point of $y(x)$ in approximation use two points.



$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

Problem: y_{n+1} on both sides of equation

Solution approximate y_{n+1} on RHS using Euler's Method

3.3.2 Improved Euler Method

1. Discretization: As for Euler Method
2. Approximation:
   $y(x_{n+1}) \approx y_{n+1}$ where

$$k_1 = f(x_n, y_n)$$
$$u_{n+1} = y_n + hk_1$$
$$k_2 = f(x_{n+1}, u_{n+1})$$
$$y_{n+1} = y_n + \frac{h}{2}[k_1 + k_2]$$

   textbook P199

Can show error in Improved Euler satisfies

$$|y(x_n) - x_n| \leq \hat{c}h^2$$

$\hat{c}$ depend on f, and interval where solution approximates

3.3.3 Tutorial 4

1. IVP $\neq$ DE

$$\frac{dP}{dt} = kP(M - P), P(0) = P_0$$

Method 1 Separable

$$\int \frac{dP}{M(M-P)} = k\int dt$$

$$\frac{1}{M}\left(\int \frac{dP}{P} + \int \frac{dP}{M-P}\right) = k\int dt$$

$$\frac{1}{M}(\ln|P| - \ln|M-P|) = kt + C_0$$

$$e^{\ln|\frac{P}{M-P}|} = e^{kMt+MC_0}$$

$$\frac{P}{M-P} = e^{kMt}C \qquad C = \pm e^{MC_0}$$

$$\implies \text{general solution (with IC) } P(0) = P_0$$

$$\implies \boxed{P(t) = \frac{MP_0}{P_0 + (M-P)e^{-kMT}}} \quad \text{solution for IVP}$$

2. $\frac{dP}{dt} + k\overbrace{P^2}^{N} = kMP$
   New function:

$$v(t) = P^{1-N} = P^{-1}$$

$$\frac{dv}{dt} = -\frac{1}{P^2}\cdot\frac{dP}{dt} = -kMP^{-1} + k$$

$$\frac{dv}{dt} = -kMv + k$$

$$\mu(t) = e^{kMt}$$

$$\frac{d}{dt}\left[e^{kMt}v\right] = ke^{kMt}$$

$$v(t) = \frac{1}{M} + Ce^{-kMt}$$

3. $\frac{dx}{dt} = (x-2)^2$ §2.2 # 7, figure 2.2.9

4. $\frac{dv}{dt} = -g(1+\frac{\rho}{g})v^2 \qquad y(0) = y_0$ §2.3 #14

$$\frac{dv}{dt} = -g - \rho v^2$$

Integrate to obtain

$$y(t) = y_0 + \frac{1}{\rho}\ln\left|\frac{\cos(C - \sqrt{\rho g})}{\cos C}\right|$$

5. Exact DE

$$\vec{D}F(x,y) = F_x dx + F_y dy$$

$\boxed{\vec{D}F(x,y) = 0}$ Solution $\implies F(x,y) = C$
check $F_x dx + F_y dy = 0 \iff F_{xy} = F_{yx} \iff N_y = M_x \implies \exists F(x,y)$

$$F(x,y) = \int F_x dx + f(y)$$

To find $f(y)$, use $F_y = M$

3.4 (Advanced Topic)

Proof of Existence and Uniqueness Theorem: Text Reference - Appendix A

Existence and Uniqueness Theorem
Suppose $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on a rectangle $R = \{(x,y)|a < x < b, c < y < d\}$ such that $(x_0, y_0) \in \mathbb{R}$.

Then there is an open interval I with $x_0 \in I \subset (a,b)$ such that

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0 \quad (1)$$

has a unique solution defined on I.

Lemma Let $f(x,y)$ be continuous on R as in Existence and Uniqueness Theorem and I be an open interval containing x_0. $y(x)$ is a solution of (1) if and only if $y(x)$ is an continuous function satisfying

$$y(x) = y_0 + \int_x^{x_0} f(t, y(t))dt \qquad (2)$$

Proof
(1) $\implies$ (2) Since $y(x)$ is a solution of (1) on I, it is continuous on I, hence $f(x, y(x))$ is a continuous function of x on I. Then by FTC the function $F(x) = \int_{x_0}^{x} f(t, y(t))dt$ satisfies $\frac{dF}{dx} = f(x, y(x))$

Thus F and y are both antiderivative of $f(x, y(x))$, so they must differ at most by a constant:

$$y(x) = F(x) + C$$

Using $y(x_0) = y_0$, and $F(x_0) = 0$ we have $C = y_0$.

$$y(x) = \int_x^{x_0} f(t, y(t))dt + y_0$$

(2) $\implies$ (1) Suppose $y(x)$ is a continuous function satisfying (2) on I. Then by FTC we have

$$(a) \qquad \frac{dy}{dx} = f(x, y(x)) \text{on } I$$

Also

$$(b) \qquad y(x_0) = y_0 + \int_{x_0}^{x_0} f(t, y(t))dt$$
$$= y_0$$

These two (a) (b) implies $y(x)$ satisfies (1) on I. □

3.4.1 Method of Proof of Existence and Uniqueness Theorem due to Émile Picard (1856-1941)

1. Find a sequence of functions $\{y_n(x)\}_{n=0}^{\infty}$ that approximate a solution of (2)
2. Show the sequence converges
3. Show the limit of the sequence satisfies eq.(2)
4. Show this is the only solution of (2)

Outline of steps (1) - (3)

1. Use
$$y_0(x) = y_0, x \in (a,b)$$
$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt$$
$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))dt$$
In general
$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt$$
This is called Picard Iteration or Method of Successive Approximations
Can show that all the $y_n(x)$ are continuous on some $I_0 \subset (a,b)$

2. Show that there is an interval $I_1 \subset I_0$ with $x_0 \in I_1$ such that $\lim_{n\to\infty} y_n(x) = \phi(x)$ uniquely on I_1
That is, given $\varepsilon > 0$ there is $N > 0$ such that
$$n > N \implies |y_n(x) - \phi(x)| < \varepsilon \qquad \forall x \in I_1$$

3. Show that there exists $I \subset I_1$ with $x_0 \in I$ such that
$$\int_{x_0}^{x} f(t, y_n(t))dt \qquad \text{converges uniformly to } \int_{x_0}^{x} f(t, \phi(t))dt \text{ on } I$$
Then we have
$$\lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f(t, y_{n-1}(x))dt$$
$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(x))dt$$
Thus $\phi(x)$ is a solution of (1) by the Lemma.

Example $\frac{dy}{dx} = y\cos x, \quad y(0) = y_0$
$f(x,y) = y\cos x \quad f, f_y$ are continuous on $\mathbb{R}^2$
We will work through the steps of the proof for this problem

1. Method of Successive Approximations
$$y_n(x) = y_0 + \int_0^x y_{n-1}(t)\cos t \; dt$$

$$y_0(x) = y_0, x \in \mathbb{R}$$
$$y_1(x) = y_0 + \int_0^x y_0 \cos t \; dt = y_0(1 + \sin x), x \in \mathbb{R}$$
$$y_2(x) = y_0 + \int_0^x y_1 \cos t \; dt = y_0 + \int_0^x y_0(1 + \sin t)\cos t \; dt$$
$$y_2(x) = y_0(1 + \sin x + \frac{1}{2}\sin^2 x)$$
Can show by induction
$$y_n(x) = y_0 \sum_{k=0}^{n} \frac{\sin^k x}{k!}$$

2. Note that $\sum_{k=0}^{n} \frac{\sin^k x_0}{k!}$ is the n^{th} partial sum of the series

$$\sum_{k=0}^{\infty} \frac{\sin^k x}{k!}$$

The terms in the series satisfy

$$\left|\frac{\sin^k x}{k!}\right| \leq \frac{1}{k!} \qquad \forall x \in \mathbb{R}$$

and $\sum_{k=0}^{\infty} \frac{1}{k!}$ is a convergent series of real numbers. It follows (by the Comparison Test) the series $\sum_{k=0}^{\infty} \frac{\sin^k x}{k!}$ converges uniformly $\implies \{y_n(x)\}_{n=0}^{\infty}$

3. Let $\lim_{n\to\infty} y_n(x) = \phi(x)$

Show that $\phi(x)$ is a solution of the DE

- Show that $\int_0^x y_n(x) \cos t \; dt$ converges uniformly

Let $\alpha > 0$. Given $\varepsilon > 0$, there is $N > 0$ such that

$$n > N \implies |y_n(x) - \phi(x)| < \frac{\varepsilon}{\alpha}, \quad \forall x \in \mathbb{R}$$

Let $I = (-\alpha, \alpha)$ and assume $x \in I, x > 0, n > N$
Then

$$\begin{aligned}
\left|\int_0^x y_n(t) \cos t \; dt - \int_0^x \phi(t) \cos t \; dt\right| &= \left|\int_0^x [y_n(t) - \phi(t)] \cos t \; dt\right| \\
&\leq \int_0^x |y_n(t) - \phi(t)| \cdot |\cos t| dt \\
&< \int_0^x \frac{\varepsilon}{\alpha} \cdot 1 \; dt \\
&\leq \frac{\varepsilon}{\alpha} \int_0^{\alpha} dt \\
&= \varepsilon
\end{aligned}$$

Can do a similar analysis for $x < 0$
Conclusion

$$\int_0^x y_n(t) \cos t \; dt \to \int_0^x \phi(x) \cos t \; dt$$

uniformly on $I = (-\alpha, \alpha)$
Take limits in approximation scheme

$$\lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} y_0 + \lim_{n\to\infty} \int_0^x y_{n-1}(t) \cos t \; dt$$

$$\phi(x) = y_0 + \int_0^x \phi(t) \cos t \; dt$$

$\implies \phi(x)$ is a solution of the IVP by Lemma from last class

3.4.2 Tutorial

Sec 2.5 #7 $\;y' = -3x^2 y, y(0) = 3 \quad y = 3e^{-x^3}$ calculate error

Method Improved Euler

- predictor (1st Euler)
- corrector

Text: 499, Appendix Ex.2

4

Dimensional Analysis

Not in textbook... See notes posted on Course webpage.

When formulating and solving models of physical systems, we need to know the dimensions of the physical quantities in the model, i.e. what the quantity represents physically.
(Linked to the units - an abstract way of representing this information)

Fundamental Physical Quantities

- mass - dimensions denoted $\mathfrak{M}$[1]
- length - dimensions denoted $\mathscr{L}$
- time - dimensions denoted τ
- [ ] - to mean the dimensions of

Let y be the position of an object and t be time

$$[y] = \mathscr{L} \quad [t] = \tau \quad \left[\frac{dy}{dt}\right] = \mathscr{L}\tau^{-1}$$

Fundamental Principles

1. One can only add, subtract or equate physical quantities with the same physical dimensions
2. Quantities with different dimensions may be combined by multiplication with dimensions given by

$$[AB] = [A][B], \quad \left[\frac{A}{B}\right] = \frac{[A]}{[B]}$$

Example Falling body with air resistance

$$m\frac{dv}{dt} = mg - kv$$

Find the dimensions of the constant k

Principle 1 $\left[m\frac{dv}{dt}\right] = [mg] = [kv]$

[1] I will match the fonts posted on learn later on, which is \mathcal

Principle 2 $[m]\left[\frac{dv}{dt}\right] = [m][g] = [k][v]$

$$\mathfrak{M}\mathscr{L}\tau^{-2} = \mathfrak{M}\mathscr{L}\tau^{-2} = [k]\mathscr{L}\tau^{-1}$$

$$\implies [k] = \mathfrak{M}\tau^{-1}$$

Dimensionless variables

Are variables formed by rescaling each physical variable by a combination of constants with the same dimensions

Procedure

1. Find dimensions of all variables and constants
2. For each variable find a combination of constants with the same dimensions
3. Define new variables by dividing the physical variables by the corresponding combination of constants

Example Logistic growth model

$$\frac{dP}{dt} = kP(M-P), \qquad P(0) = P_0$$

1.
   - Dimensions of variables
     $[P] = \mathcal{P}$ (population size)
     $[t] = \tau$
   - Dimensions of constants
     $[M] = \mathcal{P}$
     $[P_0] = \mathcal{P}$

   Use analysis as in previous example to find

$$\left[\frac{dP}{dt}\right] = [kPM] \implies \mathcal{P}\tau^{-1} = [k]\mathcal{P}^2 \implies [k] = \mathcal{P}^{-1}\tau^{-1}$$

2. Both M and P_0 have same dimensions as P

$$\left[\frac{1}{kM}\right] = \frac{1}{[k][M]} = \frac{1}{(\mathcal{P}^{-1}\tau^{-1})\mathcal{P}} = \tau$$

3. Let $y = \dfrac{P}{M}, \tau = \dfrac{t}{1/kM} = t\ kM$

$$[y] = \frac{[P]}{[M]} = \frac{\mathcal{P}}{\mathcal{P}} = 1 \implies \text{dimensionless}$$

   Make the change of variables in the model
   LHS:

$$\frac{dP}{dt} = \frac{d}{dt}(My) = M\frac{dy}{dt} = M\frac{d\tau}{dt}\frac{dy}{d\tau} = M^2k\frac{dy}{d\tau}$$

   RHS:

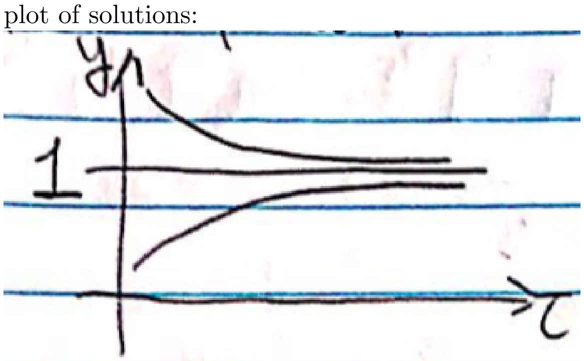
$$kP(M-P) = kMy(M-My) = kM^2y(1-y)$$

   Put together LHS = RHS

$$kM^2\frac{dy}{d\tau} = kM^2y(1-y) = \frac{dy}{d\tau} = y(1-y), \qquad y(0) = y_0$$

where $y_0 = \frac{P_0}{M}$
Solution for $y(\tau)$. $\quad y(\tau) = \dfrac{y_0}{y_0 + (1 - y_0)e^{-\tau}}$

plot of solutions:



Important constant for determining the behaviour is $y_0 = \frac{P_0}{M}$

4.1 Buckingham-π Theorem

Let $\{Q_1, Q_2, \ldots, Q_n\}$ be the set of all physical quantities (variables and constants) relevant to a particular problem. Suppose there is one and only one $\underbrace{\text{dimensionally homogeneous}}_{\text{satisfies property 1}}$ relationship between the Q_i:

$$Q_n = f(Q_1, Q_2, \ldots, Q_{n-1}) \tag{4.1}$$

where f is continuous and differentiable with respect to its variables.

Suppose there are r independent fundamental physical dimensions in the system. Then (4.1) is equivalent to

$$\pi_k = h(\pi_1, \pi_2, \ldots, \pi_{k-1}) \tag{4.2}$$

where $k = n - r$ and each π_j is a dimensionless quantity of the form

$$\pi_j = Q_1^{a_{1j}} Q_2^{a_{2j}} \ldots Q_n^{a_{nj}} \tag{4.3}$$

Proof see online notes. Relies on linear algebra □

Example Falling body with air resistance. (air resistance proportional to velocity)
Physical quantities: t, v, m, v_0, g, k
Fundamental dimensions: $\mathcal{T}, \mathcal{L}, \mathcal{M}$

Possible dimensionless quantities:

$$\pi = t^{a_1} v^{a_2} m^{a_3} v_0^{a_4} g^{a_5} k^{a_6}$$

Take dimensions of both sides, assume π is dimensionless

$$[\pi] = [t]^{a_1}[v]^{a_2}[m]^{a_3}[v_0]^{a_4}[g]^{a_5}[k]^{a_6}$$
$$1 = \mathcal{T}^{a_1}(\mathcal{L}\mathcal{T}^{-1})^{a_2}\mathcal{M}^{a_3}(\mathcal{L}\mathcal{T}^{-1})^{a_4}(\mathcal{L}\mathcal{T}^{-2})^{a_5}(\mathcal{M}\mathcal{T}^{-1})^{a_6}$$

Collet terms

$$\mathcal{T}^0\mathcal{L}^0\mathcal{M}^0 = \mathcal{T}^{a_1 - a_2 - a_4 - 2a_5 - a_6}\mathcal{L}^{a_2 + a_4 + a_5}\mathcal{M}^{a_3 + a_6}$$

Equate powers

$$a_1 - a_2 - a_4 - 2a_5 - a_6 = 0$$
$$a_2 + a_4 + a_5 = 0$$
$$a_3 + a_6 = 0$$

Rewrite in matrix form

$$\begin{bmatrix} 1 & -1 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix has rank 3 (3 linearly independent columns). There are 4 linear independent solutions of the problem.
Solve for 3 of the A_j in terms of the others

$$\begin{aligned} a_3 &= -a_6 \\ a_2 &= -a_4 - a_5 \\ a_1 &= a_2 + a_4 + 2a_5 + a_6 = a_5 + a_6 \end{aligned}$$

Put this back into the expression for π

$$\begin{aligned} \pi &= t^{a_1} v^{a_2} m^{a_3} v_0^{a_4} g^{a_5} k^{a_6} \\ &= t^{a_5+a_6} v^{-a_4-a_5} m^{-a_6} v_0^{a_4} g^{a_5} k^{a_6} \\ &= \left(\frac{v_0}{v}\right)^{a_4} \left(\frac{gt}{v}\right)^{a_5} \left(\frac{kt}{m}\right)^{a_6} \end{aligned}$$

where a_4, a_5, a_6 are arbitrary
Can choose values of a_4, a_5, a_6 to get dimensionless quantities

1. $a_4 = -1, a_5 = 0, a_6 = 0$ $\quad \pi_1 = \frac{v}{v_0} \rightarrow$ dimensionless for v
2. $a_4 = 0, a_5 = 0, a_6 = 1$ $\quad \pi_2 = \frac{kt}{m} \rightarrow$ dimensionless for t
3. $a_4 = -1, a_5 = 1, a_6 = -1$ $\quad \pi_3 = \frac{mg}{v_0} \rightarrow$ dimensionless for constant

The $\pi-$Theorem tells us the relationship $v = f(t, m, g, k, v_0)$ is equivalent to

$$\pi_1 = h(\pi_2, \pi_3)$$

We can use the dimensionless variables to rewrite the model

$$m\frac{dv}{dt} = mg - kv, \qquad v(0) = v_0$$

$$\frac{dv}{dt} = g - \frac{k}{m}v, \qquad v(0) = v_0$$

Let $w = \frac{v}{v_0}, \tau = \frac{kt}{m}$. Make the change of variables: (Exercise)

$$\frac{dw}{d\tau} = \underbrace{\frac{mg}{kv_0}}_{\lambda} - w, \qquad w(0) = 1$$

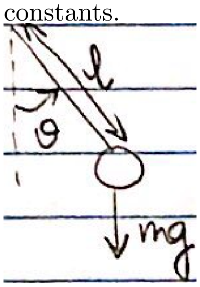
Solving the IVP for w gives

$$w(\tau) = \lambda + (1 - \lambda)e^{-\tau}$$

$w(\tau) \rightarrow \lambda$ as $\tau \rightarrow \infty$

4.2 Pendulum

Example Dimensional Analysis of a pendulum
Consider the motion of an object of mass m attached to a thin (massless) rod of length l. Let θ_0 be the initial angle the mass makes with the vertical. Assuming no friction at the pivot and no air resistances, the only force acting is gravity. In this (idealized) situation, the object will move back and forth with constant period, P. We will use the $\pi-$Theorem to deduce a relationship between P and the physical constants.



Assume that

$$P = f(\theta_0, m, l, g) \qquad (*)$$

Note that θ measured in radius is dimensionless. $\theta = \frac{s}{r}$ $\qquad [\theta] = \frac{[s]}{[r]} = \frac{\mathcal{L}}{\mathcal{L}} = 1$

Form a dimensionless quantity

$$\pi = P^{a_1}\theta_0^{a_2}m^{a_3}l^{a_4}g^{a_5}$$

Take dimensions:

$$[\pi] = [P]^{a_1}[\theta_0]^{a_2}[m]^{a_3}[l]^{a_4}[g]^{a_5}$$
$$1 = \mathcal{T}^{a_1}1^{a_2}\mathcal{M}^{a_3}\mathcal{L}^{a_4}(\mathcal{L}T^{-2})^{a_5}$$

Equate powers

$$\mathcal{M}^0\mathcal{T}^0\mathcal{L}^0 = \mathcal{T}^{a_1-2a_5}\mathcal{M}^{a_3}\mathcal{L}^{a_4+a_5}$$
$$a_1 - 2a_5 = 0$$
$$a_3 = 0$$
$$a_4 + a_5 = 0$$

$$\Longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{\text{rank 3}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix has rank 3 $\Longrightarrow$ Expect $5 - 3 = 2$ linearly independent dimensionless quantities

Solving system $\begin{cases} a_3 = 0 \\ a_1 = 2a_5 \\ a_4 = -a_5 \end{cases}$ $\qquad a_2$ is arbitrary

$$\pi = P^{2a_5}\theta_0^{a_2}m^0l^{-a_5}g^{a_5} = \theta_0^{a_2}\left(\frac{P^2g}{l}\right)^{a_5}$$

Solution

- $a_2 = 1, a_5 = 0 \qquad \pi_1 = \theta_0$
- $a_2 = 0, a_5 = \frac{1}{2} \qquad \pi_2 = P\sqrt{\frac{g}{l}}$

$\pi-$Theorem tells us $(*)$ is equivalent to $\pi_2 = h(\pi_1)$

$$P\sqrt{\frac{g}{l}} = h(\theta_0) \longrightarrow P = \sqrt{\frac{l}{g}}h(\theta_0)$$

Conclusions

- P doesn't depend on m
- P does depend on θ_0
- P is proportional to $\sqrt{\frac{l}{g}}$

4.2.1 Tutorial

1. The Schacfer model (fisheries):
   (a) Population obeys logistic, i.e. without fishing.
   (b) with fishing added, amount of caught $\sim$ (proportion to) P

$$\frac{dP}{dt} = k(M-P) - hP, \qquad P(0) = P_0$$

   $M, k, h > 0$
   $t, \tau \qquad \frac{dy}{d\tau} = (1-y) - \lambda y, \quad y(0) = y_0$

2. Moon: around Earth, period T.
   Assume T depends on M (earth), m (moon), r, G.

$$T = f(M, m, r, G)$$

$$\pi = T^{a_1} M^{a_2} m^{a_3} r^{a_4} G^{a_5}$$

$$\mathcal{T}^0 \mathcal{M}^0 \mathcal{L}^0 = \mathcal{T}^{a_1 - 2a_5} \mathcal{M}^{a_2 + a_3 - a_5} \mathcal{L}^{a_4 + 3a_5}$$

   Then ...

$$\pi_1 = T\sqrt{\frac{MG}{r^3}}(a_1 = 1, a_3 = 0) \qquad \pi_2 = \frac{m}{M}(a_1 = 0, a_3 = 1)$$

$$T = \sqrt{\frac{r^3}{MG}} h\left(\frac{m}{M}\right)$$

5 Linear Equations of Higher Order

Most general n^{th} order ODE

$$G(x, y', y'', \dots, y^{(n)}) = 0 \tag{5.1}$$

where $y' = \frac{dy}{dx}, \dots, y^{(n)} = \frac{d^n y}{dx^n}$

Example $y'' = \sin x$
Solve by integrating twice

$$\frac{d^2y}{dx^2} = \sin x \implies \frac{dy}{dx} = -\cos x + C_1 \implies y = -\sin x + C_1 x + C_2$$

Solution has two arbitrary constants

Definition A general solution of (5.1) is a solution containing n arbitrary constants that represents almost all solutions of (5.1)

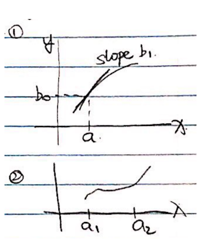
A particular solution is a solution of (5.1) with no arbitrary constants . One way to determine a particular solution from a general solution is to specify n conditions the solution must satisfy.

There can be many ways to do this.

Example $y'' = \sin x \implies y(x) = -\sin x + C_1 x + C_2$
specify conditions to determine C_1, C_2

(1) Specify values of y and y' at one value of x

(2) Specify values of y and/or y' at two values of x
$y(a_0) = b_0, y(a_1) = b_1$ or $y'(a_0) = b_0, y'(a_1) = b_1$



Type (1) are called initial conditions
A DE together with conditions of type (1) is called an initial value problem

Type (2) are called boundary conditions
A DE together with conditions of type (2) is called a boundary value problem

We will focus on IVP

Definition An n^{th} order linear ODE is an equation of the form (5.1) where G is a linear function of $y, y', y'', \ldots, y^{(n)}$.
Any n^{th} order linear ODE can be written in the form

$$\mathscr{P}_0(x)y^{(n)} + \mathscr{P}_1(x)y^{(n-1)} + \ldots + \mathscr{P}_{n-1}(x)y' + \mathscr{P}_n(x)y = F(x) \tag{5.2}$$

If $\mathscr{P}_0(x) \neq 0$ for $x \in D$, then we can rewrite (5.2) in normal form

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_{n-1}(x)y' + P_n(x)y = f(x) \tag{5.3}$$

where $P_j(x) = \frac{\mathscr{P}_j(x)}{\mathscr{P}_0(x)}, f(x) = \frac{F(x)}{\mathscr{P}_0(x)}$

Definition If $f(x) = 0$ for all $x \in D$ then (5.3) is called homogeneous otherwise it is called non-homogeneous .

Theorem (Existence and Uniqueness) Suppose that $P_1(x), P_2(x), \ldots, P_n(x)$ and $f(x)$ are continuous on the open interval I containing the point a. Then, given real numbers $b_0, b_1, \ldots, b_{n-1}$ the DE (5.3) has a unique solution on I that satisfies

$$y(a) = b_0, y'(a) = b_1, y''(a) = b_2, \ldots, y^{(n-1)}(a) = b_{n-1} \tag{5.4}$$

The associated homogeneous equation of (5.3) is

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y = 0 \tag{5.5}$$

Note that this one always has solution $y(x) = 0, x \in I$

Theorem (Principle of Superposition)
Let $y_1, y_2, \ldots, y_n$ be solutions of (5.5) on I. Then for any $c_1, c_2, \ldots, c_n \in \mathbb{R}$, the linear combination $c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$ is also a solution of (5.5) on I.

Proof (For $n = 2$), let $y_1(x), y_2(x)$ be the solutions of

$$y'' + P_1(x)y' + P_2(x)y = 0$$

Consider $\phi(x) = c_1y_1(x) + c_2y_2(x)$ where $c_1, c_2 \in \mathbb{R}$

$$\phi'(x) = c_1y_1' + c_2y_2'$$
$$\phi''(x) = c_1y_1'' + c_2y_2''$$

$$\begin{aligned}\phi'' + P_1(x)\phi' + P_2(x)\phi &= c_1y_1'' + c_2y_2'' + P_1(x)[c_1y_1' + c_2y_2'] + P_2(x)[c_1y_1 + c_2y_2] \\ &= c_1[y_1'' + P_1(x)y_1' + P_2(x)y_1] + c_2[y_2'' + P_1(x)y_2' + P_2(x)y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0\end{aligned}$$

Example $y'' + y = 0$
Easy to check that $y_1(x) = \sin x, y_2(x) = 5\sin x$.
Question: Is $c_1 \sin x + c_2 5 \sin x$ a general solution?
No. Check that $y_3 = \cos x$ is also a solution of the DE. we can't write $y_3(x) = c_1 \sin x + c_2 5 \sin x$ for any values of c_1, c_2.

Definition (Linear Independence of Functions)
The functions $f_1, f_2, \ldots, f_n$ are said to be linearly dependent on the interval I provided there exist constants $c_1, c_2, \ldots, c_n$, which are not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0, \qquad \forall x \in I$$

The functions $f_1, \ldots, f_n$ are called linearly independent on the interval I if they are not linearly dependent.

Example 1 $f_1(x) = \sin x, f_2 = 5 \sin x.$
Since $5f_1(x) - f_2(x) = 0 \quad \forall x \in \mathbb{R}$, then they are linearly dependent.

Example 2 $f_1(x) = \sin x, f_2(x) = \cos x$
Let c_1, c_2 be such that

$$(*) \qquad c_1 \sin x + c_2 \cos x = 0, \qquad \forall x \in \mathbb{R}$$

$$x = 0: \quad 0 + c_2 = 0 \implies c_2 = 0$$

$$x = \frac{\pi}{2}: \quad c_1 + 0 = 0 \implies c_1 = 0$$

We want the same values of c_1, c_2 for all $x \in \mathbb{R}$. The only way for $(*)$ to be satisfied is if $c_1 = c_2 = 0$. Thus $\sin x, \cos x$ are linearly independent on $\mathbb{R}$

Definition Suppose the functions $f_1, f_2, \ldots, f_n$ are $(n-1)$ times differentiable on some interval I. The Wronskian of the functions is

$$W(f_1, f_2, \ldots, f_n) = \det \begin{bmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{bmatrix}$$

Note W is a scalar function of x, $\qquad W : I \to \mathbb{R}$.

Lemma Let $f_1, f_2, \ldots f_n$ be $(n-1)$ times differentiable on I. If $f_1, f_2, \ldots f_n$ are linearly dependent on I, the $W(f_1, f_2, \ldots, f_n) \equiv 0$ on I

Proof (For $n = 2$)
Since f_1, f_2 are linearly dependent on I there are $c_1, c_2 \in \mathbb{R}$ which are not both zero such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \qquad \forall x \in I$$

Differentiate:

$$c_1 f_1'(x) + c_2 f_2'(x) = 0, \qquad \forall x \in I$$

For any $x \in I$, these two equations form a homogeneous linear system for c_1, c_2:

$$\begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (*)$$

We know that c_1, c_2 are not both zero.Since the linear system $(*)$ has a nontrivial solution, we must have the determinant of the coefficient matrix is zero.

$$\det \begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix} = 0 \qquad \forall x \in I$$

$W(f_1, f_2) \equiv 0$ on I □

Theorem
Let $P_1(x), \ldots, P_n(x)$ be continuous on an open interval I. Suppose that $y_1, \ldots, y_n$ are solutions of the homogeneous n^{th} linear DE (5.5) on I. Then there are two possibilities

(a) $y_1(x), \ldots, y_n$ are linearly dependent on I, and $W(y_1, \ldots, y_n) \equiv 0$ on I

(b) $y_1(x), \ldots, y_n$ are linearly independent on I and $W(y_1, \ldots, y_n) \neq 0$ for any $x \in I$

Proof

(a) Follows from the lemma

(b) for $n = 2$. Let y_1, y_2 be two linearly independent solutions on I of (5.5). Assume for contradiction that $W(y_1, y_2) = 0$ at some point $a \in I$.
Consider the linear system:

$$(*) \qquad \begin{aligned} c_1 y_1(a) + c_2 y_2(a) &= 0 \\ c_1 y_1'(a) + c_2 y_2'(a) &= 0 \end{aligned} \implies \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $W(y_1, y_2) = 0$ at a, this system has a non-trivial solution for c_1, c_2. Let $Y(x) = c_1 y_1(x) + c_2 y_2(x)$ using c_1, c_2 from the solution of $(*)$

- $Y(x)$ is a solution of (5.5) by the superposition principle
- $Y(x)$ satisfies the ICs: $\quad Y(a) = 0, Y'(a) = 0$

But the trivial (zero) solution also satisfies (5.5) and these ICs. By E/U Theorem

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) = 0 \qquad \forall x \in I$$

Since c_1, c_2 are not both zero, this means y_1, y_2 are linearly dependent on I. Contradiction. Thus $W(x) \neq 0 \qquad \forall x \in I$ □

Example $\quad y'' + y = 0$
$y_1(x) = \sin x, y_2 = 2 \sin x$ are solutions on $\mathbb{R}$.

$$W(y_1, y_2) = \det \begin{bmatrix} \sin x & 2 \sin x \\ \cos x & 2 \cos x \end{bmatrix} \equiv 0 \text{ on } \mathbb{R}$$

y_1, y_2 are linearly dependent on $\mathbb{R}$.
$y_1(x) = \sin x, y_3 = 2 \cos x$ are solutions on $\mathbb{R}$.

$$W(y_1, y_2) = \det \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} = -1 \neq 0 \text{ on } \mathbb{R}$$

y_1, y_3 are linearly independent on $\mathbb{R}$.

Theorem (General Solution for a Linear Homogeneous Equation)
Let $P_1(x), \ldots, P_n(x)$ be continuous on the open interval I. Let $y_1(x), \ldots, y_n(x)$ be n linearly independent solutions on I of (5.5). If $\phi(x)$ is any solution on I, (5.5) then there are $c_1, \ldots, c_n$ such that

$$\phi(x) = c_1 y_1(x) + \ldots + c_n y_n(x), \quad \forall x \in I$$

Proof $(n = 2)$ Let $\phi(x)$ be a solution of (5.5) on I. Let $a \in I$. Consider the linear system.

$$(*) \qquad \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \phi(a) \\ \phi'(a) \end{pmatrix}$$

Since y_1, y_2 are linearly independent on I , $W(y_1, y_2) \neq 0$ on I. Thus $\det(M) \neq 0$ and $(*)$ has a solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M^{-1} \begin{pmatrix} \phi(a) \\ \phi'(a) \end{pmatrix}$$

Using these values of c_1, c_2 define

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Then $y(x)$ satisfies the IVP on I consisting of (5.5) and $y(a) = \phi(a), y'(a) = \phi'(a)$. But $\phi(a)$ also satisfies this IVP on I. So by E/U we must have:

$$\phi(x) = y(x) = c_1 y_1(x) + c_2 y_2(x) \quad x \in I$$

In other words, given $y_1, \ldots, y_n$ linearly independent solutions of (5.5), and arbitrary constants $c_1, \ldots, c_n$

$$c_1 y_1(x) + \ldots + c_n y_n(x)$$

is a general solution of (5.5).

Theorem (General Solution for a linear non-homogeneous ODE)
Let $P_1(x), \ldots, P_n(x)$ and $f(x)$ be continuous on an open interval I. Let $y_1, \ldots, y_n$ be linearly independent solutions on I of the homogeneous DE (5.5) and $y_p(x)$ a particular solution on I of the non-homogeneous DE (5.3).
If $\phi(x)$ is any solution of (5.3) on I, then there are $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\phi(x) = c_1 y_1(x) + \ldots + c_n y_n(x) + y_p(x) \qquad \forall x \in I$$

Proof $n = 2$

$$(1)' \qquad y'' + P_1(x)y' + P_2(x)y = f(x)$$

$$(2)' \qquad y'' + P_1(x)y' + P_2(x)y = 0$$

Consider: $y_n(x) = \phi(x) - y_p(x)$, $\qquad \phi, y_p$ satisfy $(1)'$

$$\begin{aligned} y_h'' + P_1(x)y_n' + P_2(x)y_n &= \phi'' + P_1(x)\phi' + P_2(x)\phi - (y_p'' + P_1(x)y_p' + P_2(x)y_p) \\ &= f(x) - f(x) = 0 \end{aligned}$$

Thus $y_n(x)$ satisfies $(2)'$
Form the previous theorem, there are constants $c_1, c_2 \in \mathbb{R}$ such that $y_n(x) = c_1 y_1(x) + c_2 y_2(x), \forall x \in I$.
(y_1, y_2 are linearly independent solutions of $(2)'$ on I)
Thus

$$\phi(x) = y_h(x) + y_p(x) = c_1 y_1 + c_2 y_2 + y_p, \qquad \forall x \in I$$

□

Summary To find a general solution of the non-homogeneous DE (5.3) we need

(1) n linearly independent solutions of the associated homogeneous equation (5.5)

(2) a particular solution of (5.3)

We will start with (5.3) for a special cases end of midterms...

5.1 Homogeneous, linear ODEs with constant coefficients

In this case $P_1(x), \dots, P_n(x)$ are constants, so the DE can be written

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0 \tag{5.6}$$

where $a_0, \dots, a_n \in \mathbb{R}$.

What kinds of functions satisfy (5.6). There is a linear combination of y and its first n derivatives that is zero.

$y = e^{rx} \qquad y' = re^{rx} \qquad y'' = r^2 e^{rx}$

Look for solutions in the form $y = e^{rx}$, where r is TBD.

Substituting into (5.6)

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \dots + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$(a_n r^n + \dots + a_1 r + a_0) e^{rx} = 0$$

Since $e^{rx} \neq 0$ fot this equation to be satisfied, we need

$$\underbrace{a_n r^n + \dots + a_1 r + a_0}_{p(r) \quad \text{polynomial in } r} = 0 \tag{5.7}$$

Equation (5.7) is called the characteristic equation for (5.6). $p(r)$ is called the characteristic polynomial for (5.6).

Summary If r is a root of (5.7) (a zero of $p(r)$) , e^{rx} is a solution of (5.6).

Special case: $n = 2$

$$a_2 y'' + a_1 y' + a_0 y = 0 \tag{5.8}$$

Characteristic equation

$$a_2 r^2 + a_1 r + a_0 = 0 \tag{5.9}$$

Roots of (5.9): $r \pm \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$

Three cases

(1) Two real distinct roots ($a_1^2 - 4a_0 a_2 > 0$), denote by $r_+, r_- \in \mathbb{R}$.
$\implies$ two solutions of (5.8): $\quad y_1(x) = e^{r_+ x},\ y_2(x) = e^{r_- x}, \quad x \in \mathbb{R}$

(2) One real root ($a_1^2 - 4a_0 a_2 = 0$), $r_+ = r_- = r = -\frac{a_1}{2a_2} \in \mathbb{R}$
$\implies$ one solution of (5.8). $y_1(x) = e^{rx}, x \in \mathbb{R}$
$\implies$ Need a second, linearly independent solution

(3) Two complex roots ($a_1^2 - 4a_0 a_2 < 0$). $r_+, r_- \in \mathbb{C}$.
$r_\pm = \alpha \pm i\beta, \quad \alpha, \beta \in \mathbb{R}. \qquad \alpha = -\frac{a_1}{2a_2}, \beta = \frac{\sqrt{4a_0 a_2 - a_1^2}}{2a_2}$

$$e^{(\alpha \pm i\beta)x} = e^{\alpha x} e^{\pm i\beta x} = e^{\alpha x}(\cos \beta x \pm i \sin \beta x) \implies \text{ complex valued function?}$$

Theorem (Distinct Real Roots)

If the roots $r_1, \dots, r_n$ of the characteristic equation (5.7) are real and distinct and $c_1, \dots, c_n$ are arbitrary constants then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x} \qquad x \in \mathbb{R} \tag{5.10}$$

is a general solution of (5.6)

Proof $(n = 2)$
If $r_1, r_2 \in \mathbb{R}$, $r - 1 \neq r_2$ are roots of (5.9), then from our analysis above $y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x}$ are two solutions on $\mathbb{R}$ of (5.8).

$$W(y_1, y_2) = \det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{bmatrix} = (r_2 - r_1)e^{(r_1 + r_2)x} \neq 0, \forall x \in \mathbb{R}$$

$\implies y_1, y_2$ are linearly independent.
By the Theorem on general solution for homogeneous equations $c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad x \in \mathbb{R}$ us a general solution for (5.8) □

To proceed to the other cases, we'll make a brief aside.

Linking DE's to Linear Operations

Operator a mapping from function to function

$$T[f(x)] = g(x)$$

Definition The differential operator, D, is the operator defined by $D[y] = y'$. y gets mapped to its derivative.

Definition The identity operator, g, is the operator that maps a function to itself: $g[y] = y$.
Compositions of operators are defined as

$$(T_1 \circ T_2)[f] = T_1[T_2[f]]$$

Examples
$(D \circ D)[y] = D^2[y] = D[D[y]] = D[y'] = y''$
$D^k[y] = D[D[\ldots D[y] \ldots]] = y^{(k)}$

Definition An operator T is linear if for any $c_1, c_2 \in \mathbb{R}$ and any f_1, f_2 in the domain of T, $c_1 f_1 + c_2 f_2$ is in the domain of T and

$$T[c_1 f_1 + c_2 f_2] = c_1 T[f_1] + c_2 T[f_2]$$

Exercise Show D^k is a linear operator for any k.

Given $a_1, a_2 \in \mathbb{R}$ and operators T_1, T_2, we define

$$(a_1 T_1 + a_2 T_2)[y] = a_1 T_1[y] + a_2 T_2[y]$$

Example (A polynomial differential operator)
Let $a_1, a_2 \in \mathbb{R}$. For any differential function y

$$(a_1 D + a_0 g) = a_1 D[y] + a_0 g[y] = a_1 y' + a_0 y$$

Examples
Commutativity

$$(D - ag) \circ (D - bg)[y] = (D - bg) \circ (D - ag)[y]$$

Factoring

$$(D^2 - (a + b)D - abg)[y] = (D - ag) \circ (D - bg)[y]$$

Often we will write $ag[y] = a[y]$ where $a \in \mathbb{R}$

Using these ideas we may rewrite the homogeneous DE (5.6) as an operator equation

$$\begin{aligned} &a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \\ &a_n D^n[y] + \ldots + a_1 D[y] + a_0 g[y] = 0 \to \text{ function} \\ &\underbrace{(a_n D^n + \ldots + a_1 D + a_0 g)}_{P(D)}[y] = 0 \end{aligned} \tag{5.11}$$

Then solving (5.6) is equivalent to finding the functions y that are mapped to 0 by $P(D)$.
Note that $P(D)$ is a linear operator.

Example For $n = 2$, we have $a_2 y'' + a_1 y' + a_0 y = 0$.
Operator form:

$$\begin{aligned} &(a_2 D^2 + a_1 D + a_0)[y] = 0 \\ &(a_2 g) \circ (D^2 + b_1 D + b_0 g)[y] = 0 \qquad b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_0}{a_2} \\ &(a_2 g) \circ (D - r_+) \circ (D - r_-)[y] = 0 \end{aligned}$$

where $r_\pm = \dfrac{-b_1 \pm \sqrt{b_1^2 - 4b_0}}{2} = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$

Repeated Roots $r_+ = r_- = r = -\frac{a_1}{2a_2}$
Operator form of DE: $(*)$

$$a_2(D - r)^2[y] = 0$$

$$a_2(D - r)[(D - r)[y]] = 0$$

One solution is $y = e^{rx}$ since $(D - r)[e^{rx}] = re^{rx} - re^{rx} = 0$
Look for a second solution in the form $u(x)e^{rx}$.
Substitute this into $(*)$

$$\begin{aligned} a_2(D - r)[(D - r)[u(x)e^{rx}]] &= a_2(D - r)[u'e^{rx} + rue^{rx} - rue^{rx}] \\ &= a_2(D - r)[u'e^{rx}] \\ &= a_2[u''e^{rx} + u're^{rx} - u're^{rx}] \\ &= a_2 u'' e^{rx} \end{aligned}$$

In order for $u(x)e^{rx}$ to satisfy $(*)$ we need

$$a_2 u'' e^{rx} = 0 \iff u''(x) = 0 \iff u(x) = c_1 + c_2 x, \qquad c_1, c_2 \text{ constants}$$

$\Longrightarrow$ Any solution of $(*)$ is in this form $(c_1 + c_2 x)e^{rx}$, $x \in \mathbb{R}$.
Can check that e^{rx}, xe^{rx} are linearly independent functions.

Theorem (Repeated Roots)
If the characteristic equation (5.7) has a repeated root $\overline{r} \in \mathbb{R}$ of multiplicity k, then the following are k linearly independent solutions of (5.6) on $\mathbb{R}$.

$$e^{\overline{r}x}, xe^{\overline{r}x}, x^2e^{\overline{r}x}, \ldots, x^{k-1}e^{\overline{r}x}$$

Proof Since the characteristic equation as a root $\overline{r}$ of multiplicity k it may be written.

$$p(r) = q(r)(r-\overline{r})^k = 0 \qquad \text{where } q(r) \text{ is of degree } n-k$$

Thus (5.6) in operator form can be written

$$q(D)(D-\overline{r})^k[y] = 0$$

We know $e^{\overline{r}x}$ is a solution of the DE and y will be a solution if $(D-\overline{r})^k[y] = 0$
Consider $u(x)e^{rx}$. This will be a solution if

$$(D-\overline{r})^k[ue^{\overline{r}x}] = 0$$

Can show by induction that

$$(D-\overline{r})^k[ue^{\overline{r}x}] = u^{(k)}e^{\overline{r}x}$$

$ue^{\overline{r}x}$ is a solution of (5.6) if and only if $u^{(k)}e^{\overline{r}x} = 0$ if and only if $u^{(k)} = 0$ if and only if

$$u(x) = c_1 + c_2x + \ldots + c_kx^{k-1}, \qquad c_1, \ldots, c_k \in \mathbb{R}$$

Thus $c_1e^{\overline{r}x} + \ldots + c_kx^{k-1}e^{\overline{r}x}$ is a solution of (5.6) for any $c_1, \ldots, c_k \in \mathbb{R}$
$\implies e^{\overline{r}x}, xe^{\overline{r}x}, \ldots, x^{k-1}e^{\overline{r}x}$ are solutions on $\mathbb{R}$ of (5.6).
Can show that these are linearly independent. □
(See problem 29 in section 3.2 of text)

Complex roots
Recall for $n = 2$

$$a_2y'' + a_1y' + a_0y = 0$$

$$p(r) = a_2r^2 + a_1r + a_0 = 0$$

If $a_1^2 - 4a_0a_2 < 0$, the roots are complex.
$r_\pm = \alpha \pm i\beta$ where $\alpha = -\frac{a_1}{2a_2}, \beta = \frac{1}{2a_2}\sqrt{4a_0a_2 - a_1^2}$.
Note that $p(r)$ may be rewritten:

$$\begin{aligned} p(r) &= a_2(r^2 + b_1r + b_0r) \\ &= a_2(r-(\alpha+i\beta))(r-(\alpha-i\beta)) \\ &= a_2(r^2 - 2\alpha r + \alpha^2 + \beta^2) \\ &= a_2((r-\alpha)^2 + \beta^2) \end{aligned}$$

Theorem (Complex Roots)
If the characteristic equation (5.7) has a pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity 1. Then two linearly independent solutions of (5.6) on $\mathbb{R}$ are

$$y_1(x) = e^{\alpha x}\cos(\beta x), \qquad y_2(x) = e^{\alpha x}\sin(\beta x)$$

Proof In this situation the characteristic polynomial can be written $p(r) = q(r)((r-\alpha)^2 + \beta^2)$ where $q(r)$ is of degree $n-2$.
Thus the DE (5.6) in operator form is

$$q(D)((D-\alpha)^2 + \beta^2)[y] = 0$$

Consider

$$(D-\alpha)[e^{\alpha x}\cos\beta x] = \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x - \alpha e^{\alpha x}\cos\beta x = -\beta e^{\alpha x}\sin\beta x$$

Similarly $(D-\alpha)[e^{\alpha x}\sin\beta x] = \beta e^{\alpha x}\cos\beta x$

Thus

$$\begin{aligned}[(D-\alpha)^2+\beta^2][e^{\alpha x}\cos\beta x] &= (D-\alpha)^2[e^{\alpha x}\cos\beta x] + \beta^2 e^{\alpha x}\cos\beta x \\ &= (D-\alpha)[-\beta e^{\alpha x}\sin\beta x + \beta^2 e^{\alpha x}\cos\beta x] \\ &= -\beta^2 e^{\alpha x}\cos\beta x + \beta^2 e^{\alpha x}\cos\beta x = 0\end{aligned}$$

Similarly, we can show $((D-\alpha)^2+\beta^2)[e^{\alpha x}\sin\beta x] = 0$
So $e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x$ are solutions of (5.6)

Consider

$$W = \det\begin{bmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x & \alpha e^{\alpha x}\sin\beta x + \beta e^{\alpha x}\cos\beta x\end{bmatrix}$$

Thus $W(e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x) = \beta e^{2\alpha x} \neq 0$ on $\mathbb{R}$.
So $e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x$ are linearly independent on $\mathbb{R}$. □

Theorem (Repeated Complex Roots)
If the characteristic equation (5.7) has a pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity k, then the following are $2k$ linearly independent solutions of (5.6)

$$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x, \quad xe^{\alpha x}\cos\beta x, \ldots, x^{k-1}e^{\alpha x}\cos\beta x, \quad x^{k-1}e^{\alpha x}\sin\beta x$$

Proof Puts together the ideas from previous two theorems. □

Example Find the solution of the IVP $y'' - 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 2$

Solution The characteristic equation is: $r^2 - 2r + 5 = 0, \qquad (D^2 - 2D - 5g)(y) = 0.$
Roots $r = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$
From the theorem last class, two linearly independent solutions are $y_1(x) = e^x\cos 2x, \quad y_2(x) = e^x \sin 2x$
General solutions: $y(x) = c_1 e^x \cos 2x + c_2 e^x \sin 2x.$ where $c_1, c-2$ are arbitrary constants.

$$y'(x) = c_1 e^x\cos 2x - 2c_1 e^x \sin 2x + c_2 e^x \sin 2x + 2c_2 e^x \cos 2x$$

Apply IC's
$y(0) = 1 \implies 1 = c_1$
$y'(0) = 2 \implies 2 = c_1 + 2c_2$
$\implies c_1 = 1, c_2 = \frac{1}{2}$

Solution of the IVP

$$y(x) = e^x \cos 2x + \frac{1}{2} e^x \sin 2x, \quad x \in \mathbb{R}$$

Example Find a general solution of $y''' - 3y'' + 3y' - y = 0$.
Characteristic equation $r^3 - 3r^2 + 3r - 1 = 0 \implies (r-1)^3 = 0$
Roots $r = 1$, with multiplicity 3.
From Theorem last class, 3 linearly independent solution of the DE are e^x, xe^x, x^2e^x
$\implies$ General solution: $y(x) = c_1e^x + c_2xe^x + c_3x^2e^x$ where c_1, c_2, c_3 are arbitrary.

Example Find a general solution of

$$y^{(4)} - 3y'' - 4y = 0$$

Characteristic equation: $r^4 - 3r^2 - 4 = 0$
$(r^2 + 1)(r^2 - 4) = 0$
$r^2 = 4 \implies r = \pm 2$
$r^2 = -1 \implies \implies r = \pm i \implies \alpha = 0, \beta = 1$

From our Theorem from last class, 4 solutions of the DE are:

$$y_1(x) = e^{2x}, y_2(x) = e^{-2x}, y_3(x) = \cos x, y_4(x) = \sin x$$

Need to check these are linearly independent.
From Maple, $W(y_1, y_2, y_3, y_4) = -100 \neq 0 \implies$ linear independent on $\mathbb{R}$.
General solution:

$$y(x) = c_1e^{2x} + c_2e^{-2x} + c_3 \cos x + c_4 \sin x, \qquad x \in \mathbb{R}$$

where c_1, c_2, c_3, c_4 are arbitrary.

5.2 Applications

Slides on learn. (and appended here)

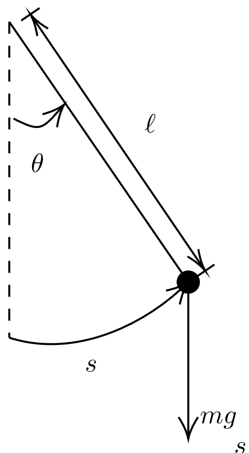
5.2.1 Mass-Spring System Model

$mx'' + cx' + kx = 0$ (constant coefficient, 2^{nd} order)

- x - displacement of object from rest position
- t - time
- $m > 0$ - mass
- $c > 0$ - damping constant
- $k > 0$ - spring constant

Initial condition $x(0) = x_0$ (initial position) $x'(0) = v_0$ (initial velocity)

5.2.2 Pendulum Model



$s = \ell\theta \qquad \theta = \dfrac{s}{\ell}$

$$m\ell\theta'' + mg\sin(\theta) = 0$$

$$\theta'' + \frac{g}{l}\sin(\theta) = 0 \qquad \text{not linear}$$

g - gravitational acceleration
ℓ - length of a string
Initial conditions: $\theta(0) = \theta_0$ (initial angle) $\qquad \theta'(0) = v_0$ (initial angular velocity)

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \qquad \theta(0) = \theta_0, \quad \theta'(0) = v_0$$

$\theta(t)$ - angle pendulum makes with vertical
This is a non-linear DE due to $\sin\theta$

Note this has the equilibrium solution: $\theta(t) = 0, t \in \mathbb{R}$
Corresponds to the pendulum hanging straight down.
If $|\theta|$ is small, then we can use the linear approximation. $\sin\theta \approx \theta$
If we put this in the model we get a linear, constant coefficient DE:

$$\theta'' + \frac{g}{\ell}\theta = 0, \qquad \theta(0) = \theta_0, \theta'(0) = v_0$$

The two applications have models in the form

$$y'' + b_1 y + b_0 y = 0$$

with mass spring: $b_1 = \frac{c}{m} > 0, \quad b_0 = \frac{k}{m} > 0$
linear pendulum: $b_1 = 0, \quad b_2 = \frac{g}{l} > 0$

Depending on the values of b_1, b_0 the systems will have different behaviour.

Free, undamped case $b_1 = 0$ (Pendulum or mass-spring with $c = 0$)
Sine $b_0 > 0$, let $b_0 = \omega_0^2, (\omega_0 > 0)$
Characteristic equation: $r^2 + \omega_0^2 = 0 \implies$ roots: $r = \pm i\omega_0$
General solution: $y(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t \qquad c_1, c_2$ arbitrary.
Rewrite (Using trig identity): $y(t) = A\cos(\omega_0 t - B) \qquad A, B$ arbitrary.
Those arbitrary constants are determined by IC's
The motion is periodic with amplitude A and period $\frac{2\pi}{\omega_0}$
This is called simple harmonic motion

Free, damped motion $(b_1, b_0 > 0)$
Characteristic equation: $r^2 + b_1 r + b_0 = 0$

- case 1 $b_1 - 4b_0 < 0$
  Complex conjugate roots $= r_\pm = \alpha \pm i\beta = -\frac{b_1}{2} + \frac{i}{2}\sqrt{b_1^2 - 4b_0}$
  General solution:
  $$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t = Ae^{\alpha t}\cos(\beta t - B)$$
  Since $b_1 > 0, \alpha < 0$, so $\lim_{t\to\infty} y(t) = 0$
  Motion is oscillatory with amplitude that decays in time
  This is called the underdamped case.

- case 2 $b_1^2 = 4b_0$ One real, repeated root $r = -\frac{b_1}{2} < 0$
  General solution $y(t) = c_1 e^{rt} + c_2 t e^{rt}$
  $\lim_{t\to\infty} y(t) = 0$ Motion is not oscillatory.
  This is called the critically damped case

- case 3 $b_1^2 > 4b_0$ Two real, distinct roots
  $$r_\pm = -\frac{b_1}{2} \pm \frac{1}{2}\sqrt{b_1^2 - 4b_0}$$
  Since $b_1 > 0$ and $b_0 > 0 \implies r_- < r_+ < 0$
  General solution: $y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t} R$
  $\lim_{t\to\infty} y(t) = 0$ Motion is not oscillatory.
  This is called the overdamped case

5.3 Non-homogeneous DE

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y = f(x) \tag{5.12}$$

Associated homogeneous DE

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y = 0 \tag{5.13}$$

General solution of (5.12):

$$y(x) = \underbrace{y_h(x)}_{\text{general solution of (5.13)}} + \underbrace{y_p(x)}_{\text{particular solution of (5.12)}}$$

We'll see two methods for finding $y_p(x)$

5.3.1 Method of Undetermined Coefficients

Applies if $P_1(x), \ldots, P_n(x)$ are constants.
In this case the DE can be written in operator form

$$\underbrace{P(D)}_{\text{characteristic polynomial}} = f(x) \qquad \longrightarrow \text{Look for } y_p(x) \text{ that is mapped from } P(D) \text{ to } f(x)$$

Idea If $f(x)$ has a finite number of derivative look for $y_p(x)$ as a linear combination of derivatives of f.
If $y_p(x)$ is a solution of (5.13) $\quad P(D)[y_p(x)] = 0$
In this case if we consider $xy_p(x)$

$$P(D)[xy_p(x)] = a_1 y_p(x)$$

Example:

Example Find the general solution of

$$y^{(4)} - 3y'' - 4y = e^{2x}\sin x + 2\cos x$$

Solution Associated homogeneous equation: $y^{(4)} - 3y'' - 4y = 0$
has general solution: $y_h(x) = c_1e^{2x} + c_2e^{-2x} + c_3\cos x + c_4\sin x$
Form for $y_p(x)$:

$$f(x) = e^{2x}\sin x + 2\cos x = f_1(x) + f_2(x)$$
$$y_p(x) = y_{p_1}(x) + y_{p_2}(x)$$
$$y_{p_1}(x) = M_1e^{2x}\cos x + M_2e^{2x}\sin x$$
$$y_{p_2}(x) = A\cos x + B\sin x$$

Since $y_{p_2}(x)$ is a solution of associated homogeneous equation.
Multiply $y_{p_2}(x)$ by x:

$$y_{p_2}(x) = Ax\cos x + Bx\sin x$$

5.3.2 Method of Variation of Parameters

Example Find a general solution of

$$y'' + 4y = \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

Solution Associated Homogeneous equation

$$y'' + 4y = 0$$

Characteristic equation: $r^2 + 4 = 0 \implies r = \pm 2i$
General solution: $y_h(x) = c_1\cos 2x + c_2\sin 2x$
Can't use Method of Undetermined coefficients due to $\sec 2x$ on RHS.

Variation of Parameters:
Assume $y_p(x) = u_1(x)\cos 2x + u_2\sin 2x$
Using the conditions we derived:

$$u_1'\cos 2x + u_2'\sin 2x = 0$$
$$u_1'(-2)\sin 2x + u_2'2\sin 2x = \sec 2x$$

Solve for u_1' and u_2'

$$u_1' = -\frac{1}{2}\tan 2x \implies u_1(x) = \frac{1}{4}\ln(\cos 2x)$$
$$u_2' = \frac{1}{2} \implies u_2 = \frac{x}{2}$$

So $y_p(x) = \frac{1}{4}\ln(\cos 2x) \quad \cos 2x + \frac{x}{2}\sin 2x$
General solution: $y(x) = y_h(x) + y_p(x)$

$$y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\ln(\cos 2x) \quad \cos 2x + \frac{x}{2}\sin 2x$$

5.4 Applications

see slides

5.4.1 Undamped Forced Motion

5.4.2 Damped Forced Motion

6 Linear Systems of DEs

Consider the mass spring model

$$mx'' + cx' + kx = F(t) \tag{$*$}$$

Introduce the variables: $\begin{matrix} x_1 = x \\ x_2 = x' \end{matrix} \implies \begin{matrix} x_1' = x' = x_2 \\ x_2' = x'' = x_1'' \end{matrix}$

Then an equivalent way of writing $(*)$ is

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + F(t) \end{cases}$$

This is a system of two first order ODEs for $x_1(t), x_2(t)$.
In general, any n^{th} order ODE can be written as a system of n first order ODEs.

6.1 Matrices and Linear Systems

Definition A first order n-dimensional linear system of ODEs is a set of n equations involving n unknown functions $x_1, x_2, \ldots, x_n$ and their first derivatives. which can be written in the form

$$\begin{aligned} x_1' &= P_{11}(t)x_1 + \ldots + P_{1n}x_n + f_1(t) \\ &\vdots \\ x_n' &= P_{n1}(t)x_1 + \ldots + P_{nn}x_n + f_n(t) \end{aligned} \tag{6.1}$$

Alternatively, we can write this:

$$\vec{x}' = P(t)\vec{x} + \vec{f}(t) \tag{6.2}$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, (\vec{x} : \mathbb{R} \to \mathbb{R}^n) \qquad P(t) = \underbrace{\begin{bmatrix} P_{11}(t) & \ldots & P_{1n}(t) \\ \vdots & & \\ P_{n1}(t) & \ldots & P_{nn}(t) \end{bmatrix}}_{\text{matrix valued function}}, (P : \mathbb{R} \to \mathbb{R}^{n\times n}) \qquad \vec{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$P(t)$ called the coefficient matrix.

Definition A solution of (6.2) on the open interval I is a function $\vec{x}$ which is differentiable on I and satisfies (6.2) on I.

Definition An initial condition for (6.2) is a specification of $\vec{x}(t)$ for a given value of t

$$\vec{x}(t_0) = x_0 \tag{6.3}$$

Theorem (Existence and Uniqueness)
Suppose the functions $P(t)$ and $\vec{f}(t)$ are continuous on the open interval I containing the point t_0. Then there is a unique solution on I to the IVP consisting of (6.2) and (6.3), for any $\vec{x}_0 \in \mathbb{R}^n$

Definition The associated homogeneous equation for (6.2) is

$$\vec{x}' = P(t)\vec{x} \tag{6.4}$$

Theorem (Principle of Superposition)
Let $\vec{x}_1, \ldots, \vec{x}_n$ be solutions of (6.4) on the open interval I. Then for any $c_1, \ldots, c_n \in \mathbb{R}$

$$\vec{x}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t)$$

is also a solution on I of (6.4)

Proof

$$\begin{aligned}\vec{x}' &= c_1\vec{x}_1' + \ldots + c_n\vec{x}_n' \\ &= c_1P(t)\vec{x}_1 + \ldots + c_nP(t)\vec{x}_n \\ &= P(t)[c_1\vec{x}_1 + \ldots + c_n\vec{x}_n] \\ &= P(t)\vec{x}\end{aligned}$$

□

Definition Let $I \subset \mathbb{R}$ and $\vec{f}_j : I \to \mathbb{R}^n$, $j = 1, \ldots, n$
The functions $\vec{f}_1, \ldots, \vec{f}_n$ are <u>linear dependent</u> on I if there are $c_1, \ldots, c_n \in \mathbb{R}$, not all of which are zero, such that $c_1\vec{f}_1(t) + \ldots + c_n\vec{f}_n(t) = 0\ \forall t \in I$. Otherwise they are <u>linear independent</u>.

Definition Let $\vec{x}_1, \ldots, \vec{x}_n$ be solutions of (6.4). Let $M = \begin{bmatrix}\vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n\end{bmatrix}$ be the matrix with j^{th} column is $\vec{x}_j$.
The wronskian of $\vec{x}_1, \ldots, \vec{x}_n$ is

$$W(\vec{x}_1, \ldots, \vec{x}_n) = \det(M) = \det\begin{bmatrix} x_{11}(t) & \ldots & x_{n1}(t) \\ \vdots & \ddots & \vdots \\ \underbrace{x_{1n}(t)}_{\vec{x}_1(t)} & \ldots & \underbrace{x_{nn}(t)}_{\vec{x}_n(t)} \end{bmatrix}$$

Theorem (Wronskian of Solutions)
Suppose $\vec{x}_1, \ldots, \vec{x}_n$ are n solutions of (6.4) on an open interval I where $P(t)$ is continuous, then there are two possibilities:

(1) $\vec{x}_1, \ldots, \vec{x}_n$ are linearly dependent on I and $W(\vec{x}_1, \ldots, \vec{x}_n) \equiv 0$ on I.

(2) $\vec{x}_1, \ldots, \vec{x}_n$ are linearly independent on I and $W(\vec{x}_1, \ldots, \vec{x}_n) \neq 0$ on $\forall t \in I$.

Proof (Analogous to case for n^{th} order ODEs) □

Theorem (General Solution of Homogeneous Linear Systems)
Let $\vec{x}_1, \ldots, \vec{x}_n$ be n linearly independent solutions of (6.4) on an open interval I where $P(t)$ is continuous. If $\vec{x}(t)$ is any solution on I of (6.4) then there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\vec{x}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) \qquad \forall t \in I$$

Proof Let $\vec{x}(t)$ be any solution on I of (6.4). Let $t_0 \in I$, and $M(t)$ be as in the definition of the Wronskian. Since $\vec{x}_1, \ldots, \vec{x}_n$ are linearly independent on I,

$$\det(M(t_0)) = W(\vec{x}_1(t_0), \ldots, \vec{x}_n(t_0)) \neq 0$$

Thus the linear system $M(t_0)\vec{c} = \vec{x}(t_0)$ $(*)$
has a unique solution

$$\vec{c} = M^{-1}(t_0)\vec{x}(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Define $\vec{y}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t)$. This is a solution of (6.4) by the Superposition Principle and satisfies the initial condition $\vec{y}(t_0) = \vec{x}(t_0)$. But $\vec{x}(t)$ is also a solution of (6.4) satisfying the same IC. By the E/U Theorem we must have

$$\vec{x}(t) = \vec{y}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) \qquad \forall t \in I$$

□

Theorem (General Solution of Non-homogeneous Linear Systems)
Let $\vec{x}_p$ be a particular solution of (6.2) on I where $P(t)$ and $\vec{f}(t)$ are continuous. Let $\vec{x}_1, \ldots, \vec{x}_n$ be n linearly independent solution on I of (6.4).
If $\vec{x}(t)$ is any solution of (6.2) on I then there are $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\vec{x}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) + \vec{x}_p(t), \qquad \forall t \in I$$

Proof Let $\vec{x}(t)$ be any solution on I of (6.2).
Let $\vec{y}(t) = \vec{x}(t) - \vec{x}_p(t)$, then

$$\vec{y}' = \vec{x}' - \vec{x}_p' = [P(t)\vec{x} + \vec{f}] - [P\vec{x}_p + \vec{f}] = P(\vec{x} - \vec{x}_p) = P\vec{y} \quad \forall t \in I$$

So $\vec{y}(t)$ is a solution on I of (6.4). By Theorem from last class, there are $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\vec{y}(t) = c_1\vec{x}_1(t) + \ldots + \vec{c}_n\vec{x}_n(t) \qquad \forall t \in I$$

Thus

$$\vec{x}(t) = \vec{y}(t) + \vec{x}_p(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) + \vec{x}_p(t) \qquad \forall t \in I$$

□

6.2 Constant Coefficient Homogeneous Systems

6.2.1 The Eigenvalue Method

See slides

Example 1 Find a solution of the IVP

$$\vec{x}' = A\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Solution Find the eigenvalues and eigenvectors of A.

Characteristic equation: $\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = 0 \implies \lambda = 3, -1$

Eigenvector for $\lambda = 3 : \vec{v} = \binom{1}{2} \to \vec{x}_1(t) = e^{3t}\binom{1}{2}$.
$\lambda = -1, \vec{v} = \binom{1}{-2} \to \vec{x}_2(t) = e^{-t}\binom{1}{-2}$
General

$$\vec{x}(t) = c_1 e^{3t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix} c_1e^{3t} + c_2e^{-t} \\ 2c_1e^{3t} - 2c_2e^{-t}\end{pmatrix}$$

Initial condition: $\vec{x}(0) = \binom{1}{3}$

$$\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix} c_1e^{3t} + c_2e^{-t} \\ 2c_1e^{3t} - 2c_2e^{-t}\end{pmatrix} \implies c_1 = \frac{5}{4}, c_2 = -\frac{1}{4}$$

Solution of IVP: $\vec{x}(t) = \begin{pmatrix} 5/4e^{3t} - 1/4e^{-4} \\ 5/2e^{3t} + 1/2e^{-t}\end{pmatrix}$

Example 2 Find a general solution of $\vec{x}'(t) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 3\end{bmatrix}$

Solution Characteristic equation: $(\lambda-2)^2 = 0, \lambda = 2, 2$
Solve for eigenvector $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1\end{bmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Eigenvector $\vec{v} = \binom{1}{-1} \to \vec{x}_1(t) = e^{2t}\binom{1}{-1}$
2^{nd} solution $\vec{x}_2(t) = e^{2t}\vec{u} + te^{2t}\vec{v}$ where $\vec{u}$ satisfies $(A-\lambda I)\vec{u} = \vec{v}$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1\end{bmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

Choose one solution: $u = \binom{0}{-1}$
$x_2(t) = e^{2t}\binom{0}{-1} + te^{2t}\binom{1}{-1}$
General solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t), \qquad t \in \mathbb{R}$

Example Find a general solution of $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} -1 & -1 \\ 5 & -3\end{pmatrix}$

Solution Characteristic equation: $\det(A - \lambda I) = 0 \implies \lambda = -2 \pm 2i = \alpha \pm i\beta$
Find an eigenvector, $\lambda = -2 + 2i$

$$(A-\lambda I)\vec{v} = 0 \qquad \begin{pmatrix} 1-2i & -1 \\ 5 & -1-2i\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \implies v_2 = (1-2i)v_1$$

Choose $\vec{v} = \begin{pmatrix}1\\1-2i\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + i\begin{pmatrix}0\\-2\end{pmatrix} = \vec{u} + i\vec{w}$
General solution (See derivation last class)

$$\begin{aligned}\vec{x}(t) &= c_1e^{\alpha t}(\vec{u}\cos(\beta t) - \vec{w}\sin(\beta t)) + c_2e^{\alpha t}(\vec{u}\sin(\beta t) + \vec{w}\cos(\beta t)) \\ &= c_1e^{-2t}\left[\begin{pmatrix}1\\1\end{pmatrix}\cos(2t) - \begin{pmatrix}0\\-2\end{pmatrix}\sin(2t)\right] + c_2e^{-2t}\left[\begin{pmatrix}1\\1\end{pmatrix}\sin(2t) + \begin{pmatrix}0\\-2\end{pmatrix}\cos(2t)\right]\end{aligned}$$

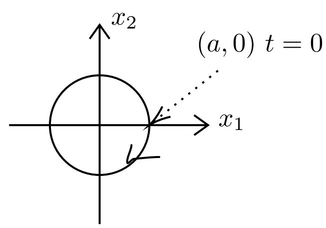
6.3 Sketching Solutions for 2-D Systems

Example $\vec{x}' = A\vec{x}, A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Can show general solution is

$$\vec{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad t \in \mathbb{R}$$

Plot of component functions separately is messy.
Suppose: $c_1 = a, \quad c_2 = 0$, then $x_1(t) = a\cos t, x_2(t) = -a\sin t, t \in \mathbb{R}$.
Think of this as parametric equation for a curve in $\mathbb{R}^2$

$$x_1^2 + x_2^2 = a^2\cos^2 t + a^2 \sin^2 t = a^2 \to \text{ a circle with radius } a$$



$t = 0 \quad x_1 = a,\ x_2 = 0$
$t = \pi/2 \quad x_1 = 0, x_2 = -a$

Put arrow to indicate how curve traversed as t increased

Suppose $c_1 = 0, c_2 = a$, similar. Can show for any values of c_1, c_2 the solution corresponding to a circle.

Example 1 Sketch the phase portrait of $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

Solution Form last lecture, A has eigenvalues $3, -1$ with eigenvectors $\binom{1}{2}, \binom{1}{-2}$.
General solution: $\vec{x}(t) = c_1 e^{3t}\binom{1}{2} + c_2 e^{-t}\binom{1}{-2}, \quad t \in \mathbb{R}$.
Special solutions: $c_1 = 0, c_2 = 0 \to \vec{x}(t) = \binom{0}{0}, t \in \mathbb{R}$.

$$c_1 \neq 0, c_2 = 0, \quad \vec{x}(t) = \begin{pmatrix} c_1 e^{3t} \\ 2c_1 e^{3t} \end{pmatrix}, t \in \mathbb{R} \implies x_2(t) = 2x_1(t), t \in \mathbb{R}$$

$$c_1 = 0, c_2 \neq 0, \quad \vec{x}(t) = \begin{pmatrix} c_2 e^{-t} \\ -2c_2 e^{-t} \end{pmatrix}, t \in \mathbb{R} \implies x_2(t) = -x_1(t), t \in \mathbb{R}$$

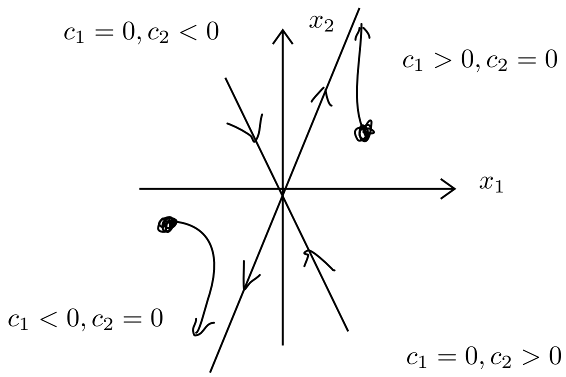
Asymptotic Behaviour $\vec{x}(t) = c_1 e^{3t}\binom{1}{2} + c_2 e^{-t}\binom{1}{-2}, \quad t \in \mathbb{R}$

- $c_1 = 0$: $\lim_{t\to\infty} \vec{x}(t) = \binom{0}{0}$ On the line $x_2 = -2x_1$ solutions move toward origin.
- $c_1 > 0, c_2 \neq 0$: $\lim_{t\to\infty} \vec{x}_1(t) = \infty, \lim_{t\to\infty} \vec{x}_2(t) = \infty$
- $c_1 < 0, c_2 \neq 0$: $\lim_{t\to\infty} \vec{x}_1(t) = -\infty, \lim_{t\to\infty} \vec{x}_2(t) = -\infty$

How do solutions behave as $t \to \infty$?

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = \lim_{t\to\infty} \frac{2c_1 e^{3t} - 2c_1 e^{-t}}{c_1 e^{3t} + c_2 e^{-t}} = 2$$

As $t \to \infty, x_2(t) \to 2x_1(t)$, $\vec{x}(t)$ has the line $x_2 = 2x_1$ as an asymptote as $t \to \infty$

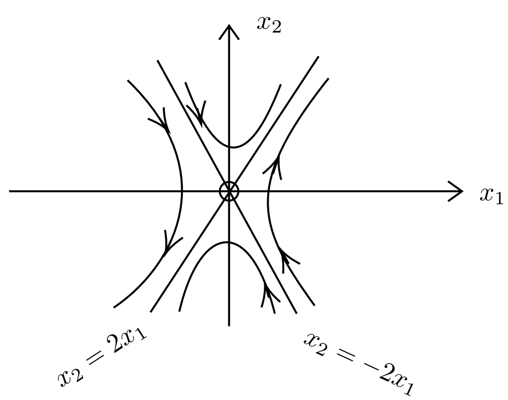


Cont'd

$$\lim_{t\to-\infty} = \frac{x_2(t)}{x_1(t)} = \lim_{t\to\infty} \frac{2c_1e^{3t} - 2c_1e^{-t}}{c_1e^{3t} + c_2e^{-t}} = -2$$

Key Points

- equilibrium point $\binom{0}{0}$ is unstable (saddle point)
- eigenvectors determine special solutions of $x_2(x_1)$
- sign of corresponding eigenvalue determines direction on special solution



Example Sketch phase potrait of $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$.

Solution A has eigenvalues 2,2, eigenvector $\binom{1}{-1}$, generalized eigenvector $\binom{0}{-1}$.
General solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} \right], \quad t \in \mathbb{R}$$

Special solutions:

c_1	c_2	
0	0	$\vec{x}(t) = \binom{0}{0}$
$\neq 0$	0	$x_2(t) = -x_1(t)$

$$c_2 = 0 \implies \lim_{t\to\infty} \vec{x}(t) = \begin{cases} \binom{+\infty}{-\infty} & c_1 > 0 \\ \binom{-\infty}{+\infty} & c_1 < 0 \end{cases}$$

$$\lim_{t\to-\infty} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 \neq 0, c_2 \neq 0 \qquad \lim_{t\to-\infty} \frac{x_2(t)}{x_1(t)} = -1$$

as $t \to -\infty$, $\vec{x}(t)$ has asymptote $x_2 = -x_1$.
Nullclines

- horizontal: $\frac{x_2}{t} = 0 \qquad x_1 + 3x_2 = 0 \implies x_2 = -\frac{x_1}{3}$
- vertical: $\frac{x_1}{t} = 0 \qquad x_1 - x_2 = 0 \implies x_1 = x_2$

equilibrium point $\binom{0}{0}$ is unstable.

Example Sketch phase potrait of $\vec{x}'(t) = \begin{pmatrix} -1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$
Eigenvalues: $-2 \pm 2i$.
General solution

$$\vec{x}(t) = c_1 e^{-2t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin(2t) \right] + c_2 e^{-2t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos(2t) \right]$$

Special solutions: $\vec{x}(t) = \binom{0}{0}$
Rewrite solution:

$$\vec{x}(t) = \kappa e^{-2t} \begin{pmatrix} \cos(2t-\delta) \\ \cos(2t-\delta) + 2\sin(2t-\delta) \end{pmatrix} \qquad \kappa = \sqrt{c_1^2 + c_2^2}, \quad \tan(\delta) = \frac{c_2}{c_1}$$

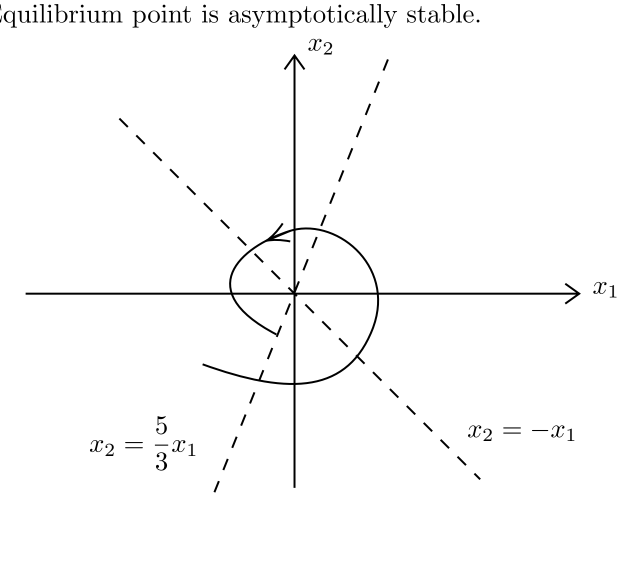
$$\lim_{t\to\infty} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad x_1(t), x_2(t) \text{ are oscillating with amplitude} \to 0 \text{ as } t \to \infty.$$

The trajectories will be spirals.

Nullclines

- horizontal: $\frac{dx_2}{dt} = 0 \implies x_2 = \frac{5}{3}x_1$
- vertical: $\frac{dx_1}{dt} = 0 \implies x_2 = -x_1$

Equilibrium point is asymptotically stable.



More details and examples in the textbook.

6.4 Fundamental Matrix

6.5 Nonhomogeneous Linear Systems

Nonhomogeneous system

$$\vec{x}' = P(t)\vec{x} + \vec{f}(t) \tag{1}$$

Associated homogeneous system

$$\vec{x}' = P(t)\vec{x} \tag{2}$$

Recall General solution of (1) is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t) \tag{3}$$

where $\vec{x}_h(t)$ is a general solution of (2), $\vec{x}_p(t)$ is a particular solution of (1). From last class

$$\vec{x}_h(t) = \Phi(t)\vec{c} \tag{4}$$

where $\Phi(t)$ is a fundamental matrix for (2), $\vec{c} \in \mathbb{R}^n$ is arbitrary .

To find $\vec{x}_p(t)$ we will use the following.

Theorem (Variation of Parameters for Linear Systems)
If $\phi(t)$ is a fundamental matrix for (2) on some open interval I where $P(t), \vec{f}(t)$ are continuous then a particular solution of (1) is

$$\vec{x}_p(t) = \Phi(t) \int \Phi^{-1} \vec{f}(t)\ dt, \quad t \in I$$

Proof Assume a solution of (1) in the form $\vec{x}_p(t) = \Phi(t)\vec{u}(t)$ where $\vec{u}(t)$ is TBD.
Then for $t \in I$

$$\vec{x}_p'(t) = \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) = P(t)\Phi(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) \quad \text{(prop3 from last class)}$$

Substitute $\vec{x}_p, \vec{x}_p'$ into (1)

$$P(t)\Phi(t)\vec{u}(t) + \Phi(t) \cdot \vec{u}'(t) = P(t)\Phi(t)\vec{u}(t) + \vec{f}(t)$$
$$\Phi(t) \cdot \vec{u}'(t) = \vec{f}(t)$$

Since Φ is invertible on I we can solve for $\vec{u}'$

$$\vec{u}'(t) = \Phi^{-1}(t)\vec{f}(t)$$

Thus we need $\vec{u}(t)$ to be an antiderivative of $\Phi^{-1}(t)\vec{f}(t)$ we write this as

$$\vec{u}(t) = \int \Phi^{-1}(t)\vec{f}(t)\ dt$$

Thus

$$\vec{x}_p(t) = \Phi(t)\vec{u}(t) = \Phi(t) \int \Phi^{-1}(t)\vec{f}(t)\ dt$$

is a solution of (1)
It follows from this Theorem and (3) that a general solution for (1) can be written as

$$\vec{x}(t) = \Phi(t)\vec{c} + \Phi(t) \int \Phi^{-1}(t)\vec{f}(t)\ dt, \quad t \in I$$

This is called the <u>variation of parameters formula</u>.

Sometimes this formula is written

$$\vec{x}(t) = \underbrace{\Phi(t)\vec{c}}_{\vec{x}_h(t)} + \underbrace{\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\vec{f}(s)\ ds,}_{\vec{x}_p(t)} \quad t \in I$$

where $t_0 \in I$.
In this case we've chosen $\vec{x}_p(t)$ so that $\vec{x}_p(t_0) = 0$

Example Find a general solution of $\vec{x}'(t) = A\vec{x} + \vec{f}(t)$ where $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ and $\vec{f}(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$.

Solution We showed in a previous lecture that

$$\vec{x}_h(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right], \quad t \in \mathbb{R}$$

A fundamental matrix for $\vec{x}' = A\vec{x}$ is

$$\Phi(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -(1+t)e^{2t} \end{pmatrix}$$

To find $\Phi^{-1}(t)$ we can use the formula. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$M^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus

$$\Phi^{-1}(t) = \begin{bmatrix} (1+t)e^{-2t} & te^{-2t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix} \implies \Phi'(t)\vec{f}(t) = \begin{pmatrix} 1+t \\ -1 \end{pmatrix}$$

$$\int \Phi'(t)\vec{f}(t)dt = \begin{pmatrix} t + \frac{t^2}{2} \\ -t \end{pmatrix} \qquad \text{leave out constants of integration}$$

$$\vec{x}_p(t) = \Phi(t)\int \Phi'(t)\vec{f}(t)dt = \textit{matrix multiplication} = \begin{pmatrix} e^{-2t}(t - \frac{t^2}{2}) \\ e^{2t}(\frac{t^2}{2}) \end{pmatrix}, \qquad t \in \mathbb{R}$$

General solution: $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$

7 Laplace Transforms

7.1 Laplace Transforms and Inverse Transforms

Definition Given a function $f(t)$ defined for all $t \geq 0$, the Laplace Transform of f is the function F defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

for all values of s for which the improper integral converges.

Recall We say the improper integral above converges if

$$\lim_{b\to\infty} \int_0^b e^{-st} f(t) dt$$

exists, otherwise we say it diverges.

Example $f(t) = 1$

$$(\text{if } s \neq 0) \quad \lim_{b\to\infty} \int_0^b e^{-st} f(t) dt = \lim_{b\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^b = \lim_{b\to\infty} \frac{1 - e^{-sb}}{s} = \begin{cases} \frac{1}{s} & \text{if } s > 0 \\ DNE & \end{cases}$$

$$\text{if } s = 0 \lim_{b\to\infty} t|_0^b = \lim_{b\to\infty} b \qquad DNE$$

Summary

$$\int_0^\infty e^{-st} dt \begin{cases} \text{converges with value } \frac{1}{s} \text{ if } s > 0 \\ \text{diverges otherwise} \end{cases}$$

Implication: $\mathcal{L}\{1\} = \frac{1}{s}$, for $s > 0$.

Example 2 Let $f(t) = e^{at}, a \in \mathbb{R}$. Assume $a \neq s$

$$\lim_{b\to\infty} \int_0^b e^{at} e^{-st} dt = \lim_{b\to\infty} \int_0^b e^{(a-s)t} dt = \lim_{b\to\infty} \frac{e^{(a-s)b} - 1}{a - s} = \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ DNE & \text{if } s < a \end{cases}$$

The case $s = a$ is the same as the case $s = 0$ in the previous example.
Conclusion: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ if $s > a$.

Example 3 Let $a \in \mathbb{C}$, i.e. $a = \alpha + i\beta$

$$e^{au} = e^{\alpha u} e^{i\beta u} = e^{\alpha u}(\cos \beta u + i \sin \beta u)$$

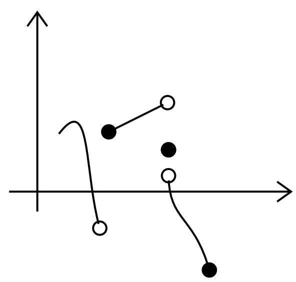
Evaluation of the integral proceeds as in the last example until the last step

$$\lim_{b\to\infty} \int_0^{\infty} e^{(\alpha+i\beta)t} e^{-st} dt = \lim_{b\to\infty} \frac{e^{(\alpha-s)b}[\cos \beta b + i \sin \beta b] - 1}{a-s} \begin{cases} \frac{1}{a-s} & \text{if } s > \alpha \\ DNE & \text{if } s \leq \alpha \end{cases}$$

To proceed further, we define conditions to guarantee that $\mathcal{L}\{f(t)\}$ exists.

Definition The function $f(t)$ is said to be piecewise continuous for $a \leq t \leq b$ provided that $[a, b]$ can be subdivided into finitely many abutting subintervals, such that

1. f is continuous in the interior of each subinterval
2. f has finite limit as t approaches each end point of each subinterval from its interior



Definition The function $f(t)$ is said to be piecewise continuous on $t \geq 0$ if it is piecewise continuous on each bounded subinterval of $[0, +\infty)$.

Example The unit step function, $u(t)$ is defined by

$$\begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

The unite step function at a is defined by

$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

Definition The function $f(t)$ is said to be of exponential order as $t \to \infty$ if there are constants $M \geq 0, c \geq 0, T \geq 0$ such that

$$|f(t)| \leq Me^{ct}, \qquad \text{for } t \geq T \qquad (2)$$

Theorem (Existence of the Laplace Transform)
If f is piecewise continuous on $t \geq 0$ and of exponential order as $t \to \infty$ with constant c in eq(2), then $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > c$.

Proof Since f is piecewise continuous on $t \geq 0$, we can find $M \geq 0$ such that (2) is satisfied with $T = 0$. i.e.

$$|f(t)| \leq Me^{ct}, \qquad \text{for } t \geq 0$$

From example 2 above $\int_0^\infty Me^{ct}e^{-st}dt$ converges if $s > c$. Thus using a comparison theorem $\int_0^\infty |f(t)e^{-st}|dt$ converges for $s > c$.

It follows that $\int_0^\infty f(t)e^{-st}dt$ converges. □

Theorem (Linearity of the Laplace Transform)
If α and β are constants and $\mathcal{L}\{f(t)\}, \mathcal{L}\{g(t)\}$, exist for $s > c$ then

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$$

Proof

$$\begin{aligned}
\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \lim_{b\to\infty} \int_0^b \left[\alpha f(t) + \beta g(t)\right] e^{-st}dt \\
&= \lim_{b\to\infty} \left[\alpha \int_0^b f(t)e^{-st}dt + \beta \int_0^b g(t)e^{-st}dt\right] \\
&= \lim_{b\to\infty} \alpha \int_0^b f(t)e^{-st}dt + \lim_{b\to\infty} \beta \int_0^b g(t)e^{-st}dt \qquad \text{since these limits exist} \\
&= \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} \text{ for } s > c
\end{aligned}$$

□

Example Find $\mathcal{L}\{\cos(kt)\}$

Solution $\cos(kt) = \frac{e^{ikt}+e^{-ikt}}{2}$

$$\mathcal{L}\{\cos(kt)\} = \mathcal{L}\left\{\frac{e^{ikt}+e^{-ikt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{ikt}\} + \frac{1}{2}\mathcal{L}\{e^{-ikt}\} = \frac{1}{2}\left[\frac{1}{s-ik} + \frac{1}{s+ik}\right] = \frac{s}{s^2+k^2} \quad \text{if } s > 0$$

Using Example 3, $\alpha = 0, \beta = k$

Similarly, we can show:

$$\mathcal{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2}, s > 0, \quad \mathcal{L}\left\{\cosh(kt)\right\} = \frac{s}{s^2-k^2}, s > k > 0, \quad \mathcal{L}\left\{\sinh(kt)\right\} = \frac{k}{s^2-k^2}, s > k > 0$$

Definition The gamma function is defined by $\Gamma(x) = \displaystyle\int_0^\infty e^{-t}t^{x-1}dt$ for $x > 0$

Properties $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(n+1) = n!$ (n a positive integer)

Example Let $a \in \mathbb{R}$, $a > -1$. Find $\mathcal{L}\left\{t^a\right\}$.

$$\begin{aligned}
&\int_0^\infty e^{-st}t^a dt \qquad \text{Let } u = st,\ t = \frac{u}{s},\ dt = \frac{1}{s}du \\
=&\int_0^\infty e^{-u}\left(\frac{u}{s}\right)^a \frac{1}{s}du \\
=&\frac{1}{s^{a+1}}\int_0^\infty e^{-u}u^0 du \qquad \text{since } t \geq 0, u > 0 \text{ if } s > 0 \\
=&\frac{1}{s^{a+1}}\Gamma(a+1) \qquad \text{for } s > 0
\end{aligned}$$

$$\implies \mathcal{L}\left\{t^a\right\} = \frac{\Gamma(a+1)}{s^{a+1}} \quad \text{for } s > 0$$

n is a positive integer $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0$

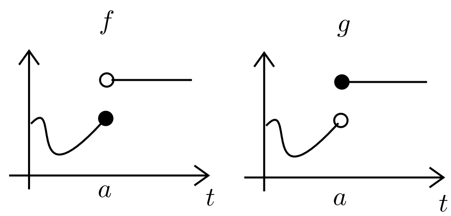
$a = -\frac{1}{2} \qquad \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \mathcal{L}\left\{t^{\frac{-1}{2}}\right\} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s}}, \quad s > 0$

Suppose we are given $F(s)$, how can we find $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$?

Theorem (Uniqueness of the Inverse Laplace Transform)
Suppose that f and g satisfy the hypothesis of the Existence Theorem for the Laplace Transform, with $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. If for some c, $F(s) = G(s)$ $\forall s > c$, then $f(t) = g(t)$ wherever on $[0, \infty)$ both functions are continuous.

Proof Requires concepts form Complex Analysis. □

Implication $f(t)$ and $g(t)$ may differ only at points of discontinuity.



$f = g$ except at a
$\Longrightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$

Suppose $g(t) = \begin{cases} f(t), & t \geq 0 \\ h(t), & t < 0 \end{cases}$ and $\mathcal{L}\{f(t)\}$ is defined for $s > c$.

$$\mathcal{L}\{g(t)\} = \int_0^\infty g(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt = \mathcal{L}\{f(t)\} \qquad s > c$$

Convention If $F(s) = \mathcal{L}\{f(t)\}$ where $f(t)$ is continuous on $t \geq 0$ we call $f(t)$ the inverse Laplace Transform of $F(s)$ and write $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Theorem (Linearity of Inverse Laplace Transform)
If $H(s) = \alpha F(s) + \beta G(s)$ for $s > c$ where $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, and f, g are continuous on $t \geq 0$m then

$$\mathcal{L}^{-1}\{H(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}, \qquad \text{where } \alpha, \beta \text{ are constants}$$

Proof Let $h(t) = \alpha f(t) + \beta g(t)$, hence $h(t)$ is continuous on $t \geq 0$.

$$\mathcal{L}\{h(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s) = H(s) \qquad \text{by linearity of } \mathcal{L}$$

Then by uniqueness,

$$\mathcal{L}^{-1}\{H(s)\} = h(t) = \alpha f(t) + \beta g(t) = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}, \qquad s > c$$

□

Example Find $\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-s-6}\right\} = \mathcal{L}^{-1}\{F(s)\}$.

Solution Apply partial fractions to $F(s)$

$$\frac{2s-3}{s^2-s-6} = \frac{A}{s-3} + \frac{B}{s+2} \implies A = \frac{3}{5}, B = \frac{7}{5}$$

Now we take inverse Laplace Transform using linearity

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}\right\} = \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$$

Example Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3+4s}\right\}$.

Solution Using partial Fractions

$$\frac{1}{s^3+4s}=\frac{A}{s}+\frac{Bs+C}{s^2+4} \implies A=\frac{1}{4}, B=-\frac{1}{4}, C=0$$

Now we take inverse Laplace Transform using linearity

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+4s}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{4s}-\frac{s}{4(s^2+4)}\right\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}=\frac{1}{4}-\frac{1}{4}\cos(2t)$$

7.2 Laplace Transforms & IVPs

Theorem (Laplace Transform of Derivatives)
Suppose $f(t)$ is continuous on $t \geq 0$ and $f'(t)$ is piecewise continuous on $t \geq 0$ and $f(t)$ is of exponential order as $t \to \infty$ (with constants c, M, T). Then $\mathcal{L}\{f'(t)\}$ exists for $s > c$

$$\mathcal{L}\{f'(t)\}=s\mathcal{L}\{f(t)\}-f(0)$$

Proof (for f' continuous) General Proof on p455-466 of text
Using IBP

$$\lim_{b\to\infty}\int_0^b f'(t)e^{-st}dt=\lim_{b\to\infty}\left[\left[f(t)e^{-st}\right]_0^b+s\int_0^b f(t)e^{-st}dt\right]=\lim_{b\to\infty}\left[\underbrace{f(b)e^{-sb}}_{\substack{\text{can show}\\ \to 0 \text{ for } s>c\\ \text{since}\\ |f(t)|\leq Me^{ct}}}-f(0)+s\underbrace{\int_0^b f(t)e^{-st}dt}_{\substack{\text{limit exists since}\\ \mathcal{L}\{f(t)\}\text{ defined for } s>c}}\right]$$

□

Corollary (Laplace Transform of Higher Derivatives)
Suppose

- $f^{(h)}$ is piecewise continuous on $t \geq 0$
- $f, f', \ldots, f^{(n-1)}$ are continuous on $t \geq 0$ and of exponential order as $t \to \infty$ (constants, M, c, T)

Then $\mathcal{L}\left\{f^{(n)}(t)\right\}$ exists for $s > c$ and

$$\mathcal{L}\left\{f^{(n)}(t)\right\}=s^n\mathcal{L}\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0)=F(s)$$

Example Use Laplace Transforms to find the solution of the IVP

$$x''-x'-6x=0, \qquad x(0)=2, x'(0)=-1$$

Solution By E/U Theorem we know a unique solution, $x(t)$, to the IVP on $\mathbb{R}$ which is twice differentiable. We assume that $\mathcal{L}\{x(t)\}$ exists and let $X(s)=\mathcal{L}\{x(t)\}$

$$\mathcal{L}\{x''-x'-6x\}=\mathcal{L}\{0\}$$

$$\mathcal{L}\{x''\}-\mathcal{L}\{x'\}-6\mathcal{L}\{x\}=0 \qquad \text{linearity}$$

$$s^2X(s)-sx(0)-x'(0)-[sX(s)-x(0)]-6X(s)=0 \quad \text{(Derivative Theorems)}$$

$$(s^2 - s - 6)X(s) - 2s + 3 = 0 \qquad \text{(using IC's)}$$
$$X(s) = \frac{2s-3}{s^2-s-6}$$

Since $x(t)$ is continuous, we use the uniqueness of the inverse.
Laplace transform to conclude.

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-s-6}\right\} = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t} \quad \text{from previous lec}$$

Solution of IVP is $x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}, t \in \mathbb{R}$

Theorem (Laplace Transform of Integrals)
If $f(t)$ is piecewise continuous on $t \geq 0$ and is of exponential order as $t \to \infty$ (with constants c, T, M) then

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s} \quad \text{for } s > c$$

equivalently:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

Proof Since f is piecewise continuous, on $t \geq 0$. $g(t) = \int_0^t f(t)dt$ is continuous on $t \geq 0$, g' is piecewise continuous on $t \geq 0$.
Further,

$$|g(t)| = \left|\int_0^t f(\tau)d\tau\right| \leq \int_0^t |f(\tau)|d\tau \leq \int_0^t Me^{c\tau}d\tau = \frac{M}{c}(e^{ct}-1) \leq \frac{M}{c}e^{ct} \qquad t \geq 0$$

So $g(t)$ is of exponential order as $t \to \infty$ and we can apply the Theorem on Laplace Transform of Derivatives.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\} \qquad \text{for } s > c$$
$$\implies \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\} \qquad \text{for } s > c$$

note the notation is :

$$\int_0^t f(\tau)d\tau$$

□

Example Solve the IVP $\qquad x''' + 4x' = 0, x(0) = x'(0) = 0, x''(0) = 1$

Solution Let $X(s) = \mathcal{L}\{x(t)\}$
Apply L.T to DE

$$\mathcal{L}\{x''' + 4x'\} = \mathcal{L}\{0\}$$
$$\mathcal{L}\{x'''\} + 4\mathcal{L}\{x'\} = 0$$
$$s^3X(s) - s^2x(0) - sx'(0) - x''(0) + 4[sX(s) - x(0)] = 0$$
$$(s^3 + 4s)X(s) = (s^2+4)x(0) + sx'(0) + x''(0) = 1$$
$$X(s) = \frac{1}{s^3+4s} = \frac{1}{s(s^2+4)}$$

To find $x(t) = \mathcal{L}^{-1}\{X(s)\}$ could use partial fractions.
Instead we'll use previous Theorem.

$$\frac{1}{s(s^2+4)} = \frac{F(s)}{s} \qquad \text{where } F(s) = \frac{1}{s^2+4}$$

Then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\} = \frac{1}{2}\sin 2t$$

By Laplace Transform of Integrals Theorem

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau = \frac{1}{2}\int_0^t \sin(2\tau)d\tau = \frac{1}{4}(1-\cos(2t))$$

7.3 Translation and Partial Fractions

Note List of partial fractions "rules" is on p. 458 - 459 in textbook.

Theorem (Translation in s)
If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, then $\mathcal{L}\{e^{at}f(t)\}$ exists for $s > a + c$ and $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.
Equivalently, $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

Proof Since $F(s)$ exists for $s > c$

$$\int_0^\infty e^{-st}f(t)dt \qquad \text{converges for } s > c$$

Replacing s by $s - a$

$$\int_0^\infty e^{-(s-a)t}f(t)dt \qquad \text{converges for } s - a > c$$

$$\Longrightarrow \underbrace{\int_0^\infty e^{-st}e^{at}f(t)dt}_{\mathcal{L}\{e^{at}f(t)\}} \qquad \text{converges for } s - a > c$$

Thus $F(s-a) = \mathcal{L}\{e^{at}f(t)\}$ for $s > a + c$. □

Immediate consequence

$$\mathcal{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}, s > a \qquad \mathcal{L}\left\{e^{at}\sin(kt)\right\} = \frac{k}{(s-a)^2+k^2}, s > a$$

$$\mathcal{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}, s > a \qquad n \text{ is postive integer}$$

Example Solve the IVP $x'' + 2x' + 5x = e^t, \qquad x(0) = 0, x'(0) = 1$

Solution Let $X(s) = \mathcal{L}\{x(t)\}$. Take Laplace Transform of DE.

$$s^2X(s) - sx(0) - x'(0) + 2[sX(s) - x(0)] + 5X(s) = \frac{1}{s-1}$$

$$(s^2+2s+5)[X(s)] = \frac{1}{s-1} + 1 \qquad \text{using ICs}$$

$$X(s) = \frac{1}{(s-1)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

Using Partial Fractions on first term

$$\frac{1}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5} \implies A = 1/8, B = -1/8, C = -3/8$$

$$\begin{aligned} X(s) &= \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{s+3}{s^2+2s+5} + \frac{1}{s^2+2s+5} \\ &= \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{s}{[(s+1)^2+4]} + \frac{1}{8}\frac{5}{[(s+1)^2+4]} \\ &= \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{s+1}{[(s+1)^2+4]} + \frac{3}{8}\frac{2}{[(s+1)^2+4]} \end{aligned}$$

Take the inverse transform

$$x(t) = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s+1}{[(s+1)^2+4]}\right\} + \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{2}{[(s+1)^2+4]}\right\} = \frac{1}{8}e^t - \frac{1}{8}e^{-t}\cos(2t) + \frac{3}{8}e^{-t}\sin(2t)$$

7.4 Derivatives, Integrals and Products of Laplace Transforms

Theorem (Differentiation of Laplace Transforms)
Suppose that $f(t)$ satisfies the conditions of the Existence Theorem and $\mathcal{L}\{f(t)\} = F(s)$ for $s > c$. Then

$$\mathcal{L}\{-tf(t)\} = \frac{dF}{ds} \qquad \text{for } s > c$$

Equivalently,

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}$$

Examples Find $\mathcal{L}\{t\sin(kt)\}$

Solution $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} = F(s)$ for $s > 0$.
Apply Theorem:

$$\mathcal{L}\{t\sin kt\} = -\frac{dF}{ds} = \frac{2ks}{(s^2+k^2)^2}$$

Application Forced, undamped mass-spring system.
$x(t)$: position of object at time t. $\omega_0 = \sqrt{\frac{k}{m}}$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t, \qquad x(0) = x'(0) = 0$$

Apply the Laplace Transform to the DE $(\mathcal{L}\{x(t)\} = X(s))$

$$(s^2+\omega_0^2)X(s) = \frac{F_0}{m}\frac{s}{s^2+\omega_0^2}$$

$$X(s) = \frac{F_0}{m}\frac{s}{(s^2+\omega_0^2)^2}$$

Take inverse transform:

$$x(t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t)$$

Corollary $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$ for $s > c$.

Theorem (Integration of Laplace Transforms)
Suppose $f(t)$ satisfies the conditions of the Existence theorem with $\mathcal{L}\{f(t)\} = F(s)$ for $s > c$, and $\lim_{t\to\infty} \dfrac{f(t)}{t}$ exists. Then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)d\sigma \qquad \text{for } s > c$$

Equivalently

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)d\sigma\right\}$$

Example Find $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2-1)^2}\right\}$

Solution We'll apply the previous theorem. Let $F(s) = \frac{2s}{(s^2-1)^2}$.
Consider

$$\lim_{b\to\infty}\int_s^b F(\sigma)d\sigma = \lim_{b\to\infty}\int_s^b \frac{2\sigma}{(\sigma^2-1)^2}d\sigma = \lim_{b\to\infty}\left[\frac{1}{s^2-1} - \frac{1}{b^2-1}\right] = \frac{1}{s^2-1}$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\} = t\mathcal{L}^{-1}\left\{\int_s^\infty \frac{2\sigma}{(\sigma^2-1)^2}d\sigma\right\} = t\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = t\sinh(t)$$

7.4.1 Products of Laplace Transforms

Consider the IVP $x'' + x = g(t), \qquad x(0) = x'(0) = 0$
Suppose L.T. of $g(t)$ exists, $\mathcal{L}\{g(t)\} = G(s)$
Let $\mathcal{L}\{x(t)\} = X(s)$ and apply L.T. to DE. $\qquad \mathcal{L}\{x''\} + \mathcal{L}\{x\} = \mathcal{L}\{g(t)\}$

$$(s^2+1)X(s) = G(s) \implies X(s) = \frac{1}{s^2+1}G(s)$$

Note that $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

Might be useful to have a way to represent $\mathcal{L}^{-1}\{X(s)\}$ in terms of $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$ and $\mathcal{L}^{-1}\{G(s)\}$.

Definition The convolution of the piecewise continuous functions f and g is defined for ≥ 0 by

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(t)g(t-\tau)d\tau$$

Exercise: show $f * g = g * f$.

Theorem (Convolution)
Suppose that $f(t)$ and $g(t)$ satisfy the conditions of the L.T. Existence Theorem and $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ for $s > c$. Then for $s > c$

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \qquad \text{and} \qquad \mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$$

Proof See text □

Back to the example $x'' + x = g(t)$, $x(0) = x'(0) = 0$

$$X(s) = \frac{1}{s^2+1}G(s), \qquad \mathcal{L}\{x(t)\} = X(s),\ \mathcal{L}\{g(t)\} = G(s),\ \mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$$

Solution of IVP: $x(t) = \mathcal{L}^{-1}\{X(s)\} = \int_0^t \sin(\tau)g(t-\tau)d\tau = \sin(t) * g(t)$
gives a representation of the solution for any $g(t)$ which satisfies conditions of L.T. existence theorem.

Suppose $g(t) = t$, $\mathcal{L}\{g(t)\} = \frac{1}{s^2}$
Solution:

$$x(t) = \int_0^t \sin(\tau)(t-\tau)d\tau = t\int_0^t \sin(\tau)d\tau - \int_0^t \tau\sin(\tau)d\tau = t\sin(t)$$

7.5 Piecewise Continuous Input Functions

Recall unit step function (Heaviside step function) at $a \geq 0$.

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

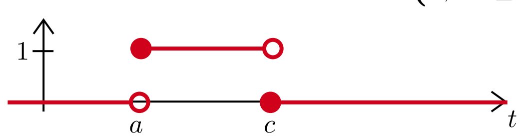
To find $\mathcal{L}\{u(t-a)\}$ use the definition

$$\lim_{b\to\infty}\int_0^b u(t-a)e^{-st}dt = \lim_{b\to\infty}\int_a^b e^{-st}dt = \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\right]_a^b = \lim_{b\to\infty}\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right] = \frac{e^{-as}}{s} \quad \text{for } s > 0$$

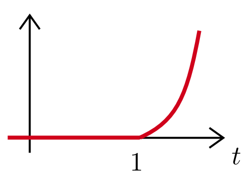
Can use $u(t-a)$ to represent piecewise defined functions.

Examples

1. $a < c$ $f(t) = u(t-a) - u(t-c) = \begin{cases} 0, & t - a \\ 1, & a \leq t < c \\ 0, & t \geq c \end{cases}$



2. $f(t) = (t-1)^2 u(t-1) = \begin{cases} 0, & t < 1 \\ (t-1)^2, & t \geq 1 \end{cases}$



Theorem (Translation in t)
Suppose $f(t)$ satisfies the conditions of the L.T existence theorem and $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \qquad \text{for } s > c$$ [1]

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = u(t-a)f(t-a)$$

[1] where textbook is incorrect

Proof Consider

$$\begin{aligned}
\lim_{b\to\infty}\int_0^b e^{-st}u(t-a)f(t-a)\,dt &= \lim_{b\to\infty}\int_a^b e^{-st}f(t-a)\,dt \qquad \text{Let } v=t-a, dv=dt\\
&= \lim_{b\to\infty}\int_0^{b-a} e^{-s(v+a)}f(v)\,dv\\
&= e^{-sa}\left[\lim_{b\to\infty}\int_0^{b-a} e^{-sv}f(v)\,dv\right] = e^{-sa}F(s) \qquad \text{for } s>c
\end{aligned}$$

$$\lim_{b\to\infty}\int_0^b e^{-st}f(t)dt = \mathcal{L}\{f(t)\} \qquad s>c$$ □

Alternate form: $\mathcal{L}\{u(t-a)g(t)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

Example Find $\mathcal{L}\{g(t)\}$ where $g(t)=\begin{cases}0, & t<2\\ t^2, & t\ge 2\end{cases}$.

Here $g(t)=u(t-2)t^2$

$$\begin{aligned}
\mathcal{L}\{g(t)\} = \mathcal{L}\left\{t^2u(t-2)\right\} &= e^{-2s}\mathcal{L}\left\{(t+2)^2\right\} \quad \text{using alternate form of Theorem}\\
&= e^{-2s}\mathcal{L}\left\{t^2+4t+4\right\}\\
&= e^{-2s}\left(\mathcal{L}\left\{t^2\right\}+4\mathcal{L}\{t\}+4\mathcal{L}\{1\}\right)\\
&= e^{-2s}\left(\frac{1}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right), \quad s>0
\end{aligned}$$

Example Solve the IVP $\quad x''+4x=u(t-1), \quad x(0)=x'(0)=0$

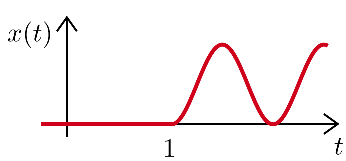
Solution Let $X(s)=\mathcal{L}\{x(t)\}$. Take L.T. of DE

$$\mathcal{L}\{x''\}+4\mathcal{L}\{x\}=\mathcal{L}\{u(t-1)\} \implies (s^2+4)X(s)=\frac{e^{-s}}{s} \longrightarrow X(s)=\frac{e^{-s}}{s(s^2+4)}$$

We saw before $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}=\frac{1}{4}(1-\cos(2t))$

Apply the previous theorem with $a=1, F(s)=\frac{1}{s(s^2+4)}$. Thus

$$x(t)=\mathcal{L}^{-1}\left\{e^{-s}F(s)\right\}=u(t-1)f(t-1)=\frac{1}{4}u(t-1)(1-\cos[2(t-1)])=\begin{cases}0, & t<1\\ \frac{1}{4}(1-\cos[2(t-1)]), & t\ge 1\end{cases}$$



7.6 Applications: Models with discontinuous forcing

- Systems with an on/off switch
  - → circuits
  - → forced mass/spring system
- Growth of population with harvesting (fishing, hunting)
  - → typically harvesting only allowed during limited time hunting season.

Example Consider an object of mass 1kg attached to a spring with spring constant 4 N/m. There is no damping but the object is attached to a motor that provides a force $F(t) = \cos(2t)$ N. If the object is at rest in its equilibrium position, when the motor is turned on for 2πseconds, find the position of the object as a function of time.

Solution

- x - position of the object (metres)
- t - time (seconds) $t = 0$ when motor is turned on.
- Model form Newton's 2^{nd} law: $mx'' + kx = F(t)$

Here m (mass) =1 kg, $k = 4$N/m (spring constant)

$$F(t) = \begin{cases} \cos(2t) & 0 \le t \le 2\pi \\ 0 & t \ge 2\pi \end{cases} = \cos(2t)\ [u(t) - u(t-2\pi)] = \cos(2t)\, u(t) - \cos(2(t-2\pi))\, u(t-2\pi)$$

Model: $x'' + 4x = \cos(2t)\, u(t) - \cos(2(t-2\pi))\, u(t-2\pi), \quad x(0) = x'(0) = 0$
Let $X(s) = \mathcal{L}\{x(t)\}$. Take Laplace Transform of DE using Derivative Theorem and initial conditions

$$s^2 X(s) + 4X(s) = \mathcal{L}\{\cos(2t)\, u(t)\} - \mathcal{L}\{\cos(2(t-2\pi))\, u(t-2\pi)\}$$

Shift in t: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

$$(s^2+4)X(s) = \mathcal{L}\{\cos(2t)\} - e^{-2\pi s}\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4} - \frac{e^{-2\pi s}s}{s^2+4}$$

$$X(s) = \frac{s}{(s^2+4)^2} - e^{-2\pi s}\frac{s}{(s^2+4)^2}$$

Recall from previous lecture: $\mathcal{L}\{t\sin(kt)\} = \dfrac{2ks}{(s^2+k^2)^2}$

$$\begin{aligned} x(t) = \mathcal{L}^{-1}\{X(s)\} &= \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{s}{(s^2+4)^2}\right\} \\ &= \frac{1}{4}t\sin 2t - \frac{1}{4}(t-2\pi)\sin(2(t-2\pi))u(t-2\pi) \\ &= \frac{1}{4}t\sin 2t - \frac{1}{4}(t-2\pi)\sin(2t)u(t-2\pi) \\ &= \frac{1}{4}\sin(2t)(1-u(t-2\pi)) + \frac{\pi}{2}\sin(2t)u(t-2\pi) \\ &= \begin{cases} \frac{t}{4}\sin 2t & 0 \le t < 2\pi \\ \frac{\pi}{2}\sin 2t & t \ge 2\pi \end{cases} \end{aligned}$$