MATH 249

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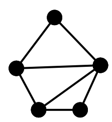
Info

- Enumeration (counting)
- Graph Theory (connecting dots)



0.1 Exam Info

- Mid: almost enumeration
- Final: almost graph theory

0.2 Office Hours

MC5028   W 2-4pm

0.3 Grading

- 15% HW biweekly (6 assignments)
- 30% Mid 9th week
- 55% Final

0.4 Note

There are several typos in the notes. If you found them, feel free to contact me!

Part I

Introduction to Enumeration

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1 Basic Principles & Linear Recurrence

1.0.1 Readings (9.7)

math239new.pdf

- Skim § 2.1 2.2
  - §2.1 Multiset
  - §2.2 birthday paradox
- Read § 4.1
- Skim § 4.2

1.1 Fibonacci numbers

$f_0 = 1,\ f_1 = 1$ and for $n \geq 2 : f_n = f_{n-1} + f_{n-2}$

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
f_n	1	1	2	3	5	8	13	21	34	55	$\cdots$

Generating Function Power series:

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

Formula for $F(x)$

$$\begin{aligned}
F(x) &= f_0 + f_1 x + \cdots + f_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n \\
&= 1 + x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \\
&= 1 + x + x \sum_{j=1}^{\infty} f_j x^j + x^2 \sum_{k=0}^{\infty} f_k x^k \\
&= 1 + x + x(F(x) - 1) + x^2 F(x)
\end{aligned}$$

$$F(x) - xF(x) - x^2 F(x) = 1$$

So $F(x) = \frac{1}{1-x-x^2}$

Geometric Series

$$\begin{aligned} G &= 1 + z + z^2 + z^3 + z^4 + \cdots \\ zG &= \quad\;\; z + z^2 + z^3 + z^4 + \cdots \end{aligned}$$

So $G = \frac{1}{1-z}$

Apply Partial fractions to F(x)
Factor the denominator as $1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$
auxiliary polynomial: $t^2 - t - 1 = (t - \alpha)(t - \beta)$
Quadratic Formula: $\alpha(\beta) = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Partial Fractions
There exists $A, B \in \mathbb{C}$ such that

$$F(x) = \frac{1}{1 - x - x^2} = \frac{1}{(1 - \alpha x)(1 - \beta x)} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}$$

Multiply by $(1 - \alpha x)(1 - \beta x)$ to clear denominator

$$1 = A(1 - \beta x) + B(1 - \alpha x) = (A + B) + (-A\beta - B\alpha)x$$

Compare coefficients

$$\begin{cases} A + B = 1 \\ A\beta + B\alpha = 0 \end{cases}$$

Which gives us

$$\begin{cases} A = \frac{\alpha}{\alpha - \beta} \\ B = \frac{\beta}{\beta - \alpha} \end{cases}$$

By calculating, we get

$$\begin{cases} A = \frac{5 + \sqrt{5}}{10} \\ B = \frac{5 - \sqrt{5}}{10} \end{cases}$$

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} f_n x^n \\ &= \frac{1}{1 - x - x^2} \\ &= \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x} \\ &= A \sum_{n=0}^{\infty} (\alpha x)^n + B \sum_{n=0}^{\infty} (\beta x)^n \quad \text{(by geometric series)} \\ &= \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n) x^n \end{aligned}$$

So we get $f_n = A\alpha^n + B\beta^n$ for all $n \in \mathbb{N}$
So,

$$f_n = \frac{5 + \sqrt{5}}{10} \left(\frac{1 + \sqrt{5}}{2} \right)^n + \frac{5 - \sqrt{5}}{10} \left(\frac{1 - \sqrt{5}}{2} \right)^n \quad \text{for all } n \in \mathbb{N}$$

1.1.1 Readings (9.10)

Read §2.1 up to Example 2.4 (or 2.5)
Read §3.1

1.2 Partial Lists

Let S be a set of size $|S| = n$.
A partial list of S of length k is a sequence $s_1 s_2 \cdots s_k$ with each $s_1 \in S$, and no two of them are equal.

Example $S = \{a,\ X,\ 4,\ ☺,\ ?\}$
Partial lists of length 2:

$aX,\ a4,\ a☺,\ a?,\ Xa,\ X4,\ X☺, \cdots$

How many partial lists of length k form a set of size n?

Theorem 2.1.3 $n(n-1)(n-2)\cdots(n-k+1)$

"Proof" How to choose a partial list $s_1 s_2 \cdots s_k$ of S with $|S| = n$?

- n choices for s_1 (any element of S)
- and $n-1$ choices for s_2 (any elements of $S \setminus \{s_1\}$
- and $\cdots$
- $\cdots$
- and $n-k+1$ choices for s_k (any element of $S \setminus \{s_1,\ s_2,\ \cdots, s_{k-1}\}$)

$n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ is the number of lists of all elements of S.

1.3 Subsets

A subset of size k of a set S of size $|S| = n$ is a collection of elements $\{s_1,\ s_2,\ \cdots, s_k\}$), no two are equal, $s_i \in S$.

Theorem 2.1.4 The number of these is $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

"Proof" Form a list of all of S: $s_1 s_2 s_3 \cdots s_n$ in one of $n!$ ways
Consider $s_1 s_2 s_3 \cdots s_k$ and $s_{k+1} s_{k+2} \cdots s_n$ separately, then $A = \{s_1,\ s_2,\ \cdots s_k\}$ is a k-element subset of S. Also $S \setminus A = \{s_{k+1},\ s_{k+2},\ \cdots s_n\}$ is an $(n-k)$-element subset of S.

$s_1 s_2 s_3 \cdots s_k$ is a list of A, and $s_{k+1} s_{k+2} \cdots s_n$ is a list of $S \setminus A$.

Let $B(n,\ k)$ be the number of k-element subsets of S with $|S| = n$.
From a list of all of S we construct:

- a k-element subset $A \subseteq S$
- a list of all of A
- a list of all of $S \setminus A$

This construction is "reversible":

It follows that $n! = B(n,\ k) \cdot k! \cdot (n-k)!$
So $B(n,\ k) = \frac{n!}{k!\,(n-k)!}$

1.4 Multisets

Let $t \geq 1$ be an integer, the number of allowed "types" of element.

A multiset of size n with elements of t types is a "set" of size n of these t types of element, but you can have more than one element of each type.

Example:
$t = 5$: types are R, G, Y, O, P.
$n = 10$

{ R, G, P, G, P, R, O, P, G, O } (might have no "Y")

The order of the elements doesn't matter.
Sort them:

{ R, R, G, G, G, O, O, P, P, P } $\leftrightarrow (m_1, m_2, \cdots m_5) = (2, 3, 0, 2, 3)$

The only information is the number of elements of each type.
Let m_i be the number of elements of type i, for $1 \leq i \leq t$

How many multisets of size n with t types of elements are there?

Theorem 2.1.5 $\binom{n+t-1}{t-1}$

"Proof" Define a one-to-one correspondence between:

- mutisets of size n with elements of t types
- subsets of size t of $\{ 1, 2, 3, \cdots, n + t - 1 \}$

Write down a sequence of $n + k - 1$ O s.

O O O O O O O O O O O O O O

Choose a $(t - 1)$ element subset of the O s by crossing them out with X s.

O O X O O O X X O O X O O O

Let m_i be the number of uncrossed O s between $(i - 1)^{\text{st}}$ and i^{th} X.

(2 , 3 , 0 , 2 , 3)

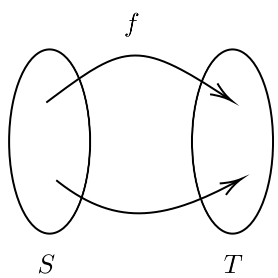
The construction is reversible.

HW #1 Revision Question A2 (Revised)

1.5 Bijection

$f : S \to T$ a function between sets S, T

- surjective:
  $\forall t \in T : \exists s \in S : f(s) = t$
- injective:
  $\forall s, s' \in S :$ if $f(s) = f(s')$ then $s = s'$
- bijective: both surjective and injective
  f is bijective if and only if $\forall t \in T, \exists$ exactly one $s \in S$ such that $f(s) = t$



A bijection has an inverse function $f^{-1} : T \to S$ defined by

$$f^{-1}(t) = s \quad \text{if and only if } f(s) = t$$

Also $(f^{-1})^{-1} = f$. The inverse function is unique.
$S \leftrightarrows T$ means there is a bijection $f : S \to T$ (S, T are equicardinal)
This is an equivalence relation.
$S \leftrightarrows T$ if and only if S, T have the same size.

1.6 Subsets & Indicator Vectors

Let P_n be the set of all subsets $S \subseteq \{1, 2, \cdots, n\}$
Let $\{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{\text{n factors}}$ Typical element $\alpha = (a_1, a_2, \cdots, a_n)$. each $a_i \in \{0,1\}$.

Bijection

$$p_n \leftrightarrows \{0,1\}^n$$

$$S \leftrightarrow \alpha = (a_1, a_2, \cdots, a_n)$$

$$\left.\begin{array}{l} S \mapsto \alpha \text{ with } a_i = \begin{cases} 0 \text{ if } i \notin S \\ 1 \text{ if } i \in S \end{cases} \quad \text{for } 1 \le i \le n \\ \{i \in \{1, 2, \cdots, n\} : a_i = 1\} = S \leftarrow\!\shortmid \alpha \end{array}\right\} \text{mutually inverse bijections}$$

Also: $|S| = a_1 + a_2 + \cdots + a_n$

1.7 Binomial Theorem

$$\sum_{S \subseteq \{1,2,\cdots,n\}} x^{|S|} = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Since $\{1, 2, \cdots, n\}$ has $\binom{n}{k}$ subsets of size k
Equivalently, (by bijection)

$$\sum_{\alpha \in \{0,1\}^n} x^{a_1+a_2+\cdots+a_n} = \sum_{a_1=0}^{1}\sum_{a_2=0}^{1}\cdots\sum_{a_n=0}^{1} x^{a_1+a_2+\cdots+a_n} \quad \text{(by Cartesian Product)}$$

$$= \sum_{a_1=0}^{1} x^{a_1} \sum_{a_2=0}^{1} x^{a_2} \cdots \sum_{a_n=0}^{1} x^{a_n}$$

$$= (\sum_{c=0}^{1} x^c)^n = (1+x)^n$$

Therefore,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

1.8 Binomial Series

Fix a positive integer t.
Let $M(t)$ be the set of all multisets with element t types, of any size.
$\mu = (m_1, m_2, \cdots, m_t)$ with each $m_i \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$
That is $M(t) = \mathbb{N}^t = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{\text{t factors}}$

$$|\mu| = m_1 + m_2 + \cdots + m_t$$

The number of multisets of size n in $M(t)$ is $\binom{n+t-1}{t-1}$
Now

$$\sum_{\mu \in M(t)} x^{|\mu|} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

by bijection. Equivalently,

$$\sum_{(m_1,\cdots,m_t)\in\mathbb{N}^t} x^{m_1+m_2+\cdots+m_t} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_t=0}^{\infty} x^{m_1+m_2+\cdots+m_t}$$

$$= \sum_{m_1=0}^{\infty} x^{m_1} \sum_{m_2=0}^{\infty} x^{m_2} \cdots \sum_{m_t=0}^{\infty} x^{m_t} = (\sum_{c=0}^{\infty} x^c)^t = \frac{1}{(1-x)^t}$$

In conclusion, for positive integer t,

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

The Binomial Series is needed for the general case of Partial Fractions.

eg

$$\frac{1-2x+2x^2}{(1+x)(1-2x)^2}$$

$$= \frac{A}{1+x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

for some $A, B, C \in \mathbb{C}$

HW Revision Question A2

1.9 Recurrence Relation

1.9.1 Readings (9.14)

Read §3.1 4.1 4.2, Skim §4.3

1.9.2 Homogeneous Linear Recurrence Relations

Quick Note

- Homogeneous: No other contributions or terms (e.g. +3 at the end)
- Linear: The first term is a linear combinations of the following terms.

Initial conditions: $c_0 = 1, c_1 = 1, c_2 = 0$

Recurrence Relation: For $n \geq 3 : c_n = 4c_{n-1} - 5c_{n-2} + 2c_{n-3}$

n	0	1	2	3	4	5	6	$\cdots$
c_n	1	0	0	2	8	22	52	$\cdots$

Let $c(x) = \sum_{n=0}^{\infty} c_n x^n$, then

$$\begin{aligned} c(x) &= 1 + 0x + 0x^2 + \sum_{n=3}^{\infty}(4c_{n-1} - 5c_{n-2} + 2c_{n-3})x^n \\ &= 1 + 4x\sum_{i=2}^{\infty} c_i x^i - 5x^2 \sum_{j=1}^{\infty} c_j x^j + 2x^3 \sum_{k=0}^{\infty} c_k x^k \\ &= 1 + 4x(c(x) - 1) - 5x^2(c(x) - 1) + 2x^3 c(x) \end{aligned}$$

Thus, by calculation

$$c(x) = \frac{1 - 4x + 5x^2}{1 - 4x + 5x^2 - 2x^3}$$

Take carefully at the denominator. If we rewrite the recurrence to the one side, we will get $c_n - 4c_{n-1} + 5c_{n-2} - 2c_{n-3} = 0$ which matches the denominator.

Initial condition (reminder): $c_0 = 1,\ c_1 = 0,\ c_2 = 0$ By convention, $c_n = 0 \quad$ if $n < 0$

$$\begin{array}{llllll} \text{For } n \geq 3: & c_n & -4c_{n-1} & +5c_{n-2} & -2c_{n-2} & = 0 \\ \text{For } n = 0 & c_0 & & & & = 1 \\ \text{For } n = 1 & c_1 & -4c_0 & & & = -4 \\ \text{For } n = 2 & c_2 & -4c_1 & +5c_0 & & = 5 \end{array}$$

The last three equations match the numerator.

1.9.3 Partial Fractions

$\dfrac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} c_n x^n$

- $P(x)$ and $Q(x)$ are polynomials.
- $\deg(P) < \deg(Q)$
- $Q(0) = 1$ (constant term of Q is 1)

Fact

$$Q(x) = (1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2}\cdots(1-\lambda_s x)^{d_s}$$

with $d_1 + d_2 + \cdots + d_s = d$ and $\lambda_1, \lambda_2, \cdots, \lambda_s$ (pairwise) distinct complex numbers

<u>Then</u>: there exists $C_i^{(j)} \in \mathbb{C}$ for $1 \le i \le s$ and $1 \le j \le d_i$ such that

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{s}\sum_{j=1}^{d_i} \frac{C_i^{(j)}}{(1-\lambda_i x)^j}$$

Proof: see chapter 4.4 (wrote by our dear instructor)

$Q(x) = 1 - 4x + 5x^2 - 2x^3$ has $Q(1) = 0$, so $1 - x$ divides $Q(x)$

$$\begin{array}{r|l} & 2x^2 - 3x + 1 \\ \hline -x+1 & -2x^3 + 5x^2 - 4x + 1 \\ & 2x^3 - 2x^2 \\ \hline & \phantom{-2x^3+{}} 3x^2 - 4x \\ & - 3x^2 + 3x \\ \hline & - x + 1 \\ & \phantom{-2x^3+5x^2-{}} x - 1 \\ \hline & \phantom{-2x^3+5x^2-4x+{}} 0 \end{array}$$

then

$$Q(x) = (1-x)(1-3x+2x^2) = (1-x)^2(1-2x)$$

$$c(x) = \frac{1-4x+5x^2}{1-4x+5x^2-2x^3} = \frac{1-4x+5x^2}{(1-x)^2(1-2x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-2x}$$

<u>**Clear denominator**</u>

$A(1-x)(1-2x) + B(1-2x) + C(1-x)^2 = 1 - 4x + 5x^2$

@ $x = 1$, $B = 2$

@ $x = \frac{1}{2}$, $\frac{C}{4} = 1 - 2 + \frac{5}{4} = \frac{1}{4}$ so $C = 1$

@ $x = 0$, $A + B + C = 1$ so $A = 2$

$$\begin{aligned} \sum_{n=0}^{\infty} c_n x^n &= \frac{2}{1-x} - \frac{2}{(1-x)^2} + \frac{1}{1-2x} \\ &= \sum_{n=0}^{\infty} \left(2 - 2\binom{n+1}{1} + 2^n\right)x^n \\ &= \sum_{n=0}^{\infty} (2^n - 2n)x^n \end{aligned}$$

So $c_n = 2^n - 2n$

1.9.4 Homogeneous Linear Recurrence Relations (cont'd)

It's in §4.2.

e.g.
$s_0 = 2, s_1 = 1, s_2 = 0, s_2 = 3$
For $n \geq 4,\ s_n - 2s_{n-1} + 2s_{n-2} - 2s_{n-3} + s_{n-4} = 0$
So,

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
c_n	2	1	0	3	6	5	4	7	10	9	$\cdots$

Let $s(x) = \sum\limits_{n=0}^{\infty} s_n x^n$.
$s(x) = 2 + x + 0x^2 + 3x^3 + \sum\limits_{n=4}^{\infty} (2s_{n-1} - 2s_{n-2} + 2s_{n-3} - s_{n-4})x^n$
By convention, say that $s_n = 0$ if $n < 0$
So,

$$\begin{aligned} s(x) &= 2 + x + 3x^3 + 2x\sum_{h=3}^{\infty} s_h x^h - 2x^2\sum_{i=2}^{\infty} s_i x^i - 2x^3\sum_{j=1}^{\infty} s_j x^j - x^4\sum_{k=0}^{\infty} s_k x^k \\ &= 2 + x + 3x^3 + 2x(s(x) - 2 - x) - 2x^2(s(x) - 2 - x) + 2x^3(s(x) - 2) - x^4 s(x) \end{aligned}$$

Then we get,

$$\begin{aligned} & s(x) - 2xs(x) + 2x^2 s(x) - 2x^3 s(x) + x^4 s(x) \\ = {} & 2 + x + 3x^3 - 4x - 2x^2 + 4x^2 + 2x^3 - 4x^3 \\ = {} & 2 - 3x - 2x^2 + x^3 \end{aligned}$$

So $s(x) = \dfrac{2 - 3x + 2x^2 + x^3}{1 - 2x + 2x^2 - 2x^3 + 4}$

<u>**Partial Fractions**</u> Factor the denominator to get inverse roots
$(1-x)$ is a factor...
then, $(1-x)^2(1+x^2) = (1-x)^2(1+ix)(1-ix)$
then, $s(x) = \dfrac{A}{1-x} + \dfrac{B}{(1-x)^2} + \dfrac{C}{1-ix} + \dfrac{D}{1+ix}$
then, $2 - 3x + 2x^2 + x^3 = A(1-x)(1+x^2) + B(1+x^2) + C(1-x)^2(1+ix) + D(1-x)^2(1-ix)$

@ $x = 1$, $\quad 2 = 2 - 3 + 2 + 1 \quad$ so $B = 2$

@ $x = i$, $\quad 2 - 3i - 2 - i = -4i = D(1-i)^2 2 = -4i\ D \quad$ so $D = 1$

@ $x = -i$, $\quad 2 + 3i - 2 + i = 4i = C(1+i)^2 2 = 4i\ C \quad$ so $C = 1$

@ $x = 0$, $\quad 2 - A + B + C + D = A + 3 \quad$ so $A = -1$

So $\quad s(x) = \dfrac{-1}{1-x} + \dfrac{1}{(1-x)^2} + \dfrac{1}{1-ix} + \dfrac{1}{1+ix}$

$$\begin{aligned} s(x) &= \sum_{n=0}^{\infty} \left(-1 + \binom{n+1}{1} + i^n + (-i)^n\right)x^n \\ &= \sum_{n=0}^{\infty} (n + i^n + (-i)^n)x^n \end{aligned}$$

So $s_n = n + i^n + (-i)^n \quad$ for all $n \in \mathbb{N}$
<u>Modulo 4</u>

$$s_n = \begin{cases} n+2 & \text{if } n \equiv 0 \pmod 4 \\ n & \text{if } n \equiv 1,3 \pmod 4 \\ n-2 & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

1.9.5 General Case

Initial conditions $g_0, g_1, g_2, \cdots, g_{N-1}$
Recurrence
$\sum_{i=0}^{d} a_i g_{n-i} = 0$ for all $n \geq N$ with $a_0 = 1$

$Q(x) = \sum_{i=0}^{d} a_i x^j$ (d is the degree of the recurrence)

Generating Function $G(x) = \sum_{n=0}^{\infty} g_n x^n = \frac{P(x)}{Q(x)}$ with this denominator $\uparrow Q(x)$

Numerator $p(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1} x^{N-1}$
in which $b_n = \sum_{i=0}^{d} a_i g_{n-i}$ for all $n \in \mathbb{N}$
with convention that $g_n = 0$ if $n < 0$

1.9.6 General Cases Cont'd

Sequence: $g_0, g_1, g_2, g_3, \ldots$

defined by
Initial Conditions: $g_0, g_1, g_2, g_3, \ldots, g_{N-1}$ have given values.

HLRR: For $n \geq N$,

$$g_n = -(a_1 g_{n-1} + a_2 g_{n-2} + \ldots + a_d g_{n-d}) \qquad (*)$$

for some numbers $a_0 = 1, a_1, a_2, \ldots, a_d$

Convention: $g_n = 0$ if $n < 0$

Generating Function: $G(x) = g_0 + g_1 x + g_2 x^2 + \ldots = \sum_{n=0}^{\infty} g_n x^n$

Polynomial: $a_0 + a_1 x + a_2 x^2 + \ldots + a_d x^d = Q(x)$ records the HLRR (remember $a_0 = 1$)

$$\begin{aligned} G(x) \cdot Q(x) &= \sum_{n=0}^{\infty} g_n x^n \cdot \sum_{j=0}^{d} a_j x^j \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{d} g_n a_j x^{n+j} \\ &= \sum_{m=0}^{\infty} (\sum_{j=0}^{d} a_j g_{m-j}) x^m \end{aligned}$$

Note: this makes sense even if $m < d$, since $g_n = 0$ if $n < 0$

For all $m \in \mathbb{N}$, the coefficient of x^m is $G(x)Q(x)$ is

$$b_m = \sum_{j=0}^{d} a_j g_{m-j}$$

Note: From $(*)$ we have $\sum_{j=0}^{d} a_j g_{m-j} = b_m = 0$ for all $m \geq N$.
So $G(x)Q(x) = P(x)$ is a polynomial of degree at most $N-1$

Thus $G(x) = \dfrac{P(x)}{Q(x)}$

The calculation can be done in the reverse direction.

e.g.// $c_0 = 1, c_1 = -1, c_2 = 3,$ for $n \geq 3, c_n = 2c_{n-1} - c_{n-3}$

$$c_n - 2c_{n-1} + c_{n-3} = \begin{cases} 1, & \text{for } n = 0 \\ -1-2 = -3, & \text{for } n = 1 \\ 3 - 2(-1) = 5, & \text{for } n = 2 \\ 0, & \text{for } n \geq 3 \end{cases}$$

So $C(x) = \sum_{n=0}^{\infty} c_n x^n = \dfrac{1 - 3x + 5x^2}{1 - 2x + x^3}$

e.g.2 Let $H(x) = \sum_{n=0}^{\infty} x^n = \dfrac{1 - x^2 + 2x^3 - 4x^2}{1 - 2x + 3x^2 - x^3}$
Then

$$h_n - 2h_{n-1} + 3h_{n-2} - h_{n-3} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n = 1 \\ -1, & \text{if } n = 2 \\ 2, & \text{if } n = 3 \\ -4, & \text{if } n = 4 \\ 0, & \text{if } n \geq 5 \end{cases}$$

HLRR $h_n = 2h_{n-1} - 3h_{n-2} + h_{n-3}$ for $n \geq 5$

Initial Conditions
$h_0 = 1, h_1 - 2h_0 = 0$, so $h_1 = 2$
$h_2 - 2h_1 + 3h_0 = -1$, so $h_2 = 0$
$h_3 - 2h_2 + 3h_1 - h_0 = \ldots$

Next class: Let g_n be the number of sequences

$r = (c_1, \ldots, c_k)$ with $k \in \mathbb{N}$ and each $c_i \in \mathbb{N}$ with each $c_i \geq 2$ and $c_1 + \ldots + c_k = n$

For $n \in \mathbb{N}$, what is g_n?

2 Compositions

A composition is a sequence $\gamma = (c_1, c_2, \ldots, c_k)$ in which each $c_i \in \mathbb{P} = \{1, 2, \ldots\}$ is a positive integer, and $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Each c_i is called a part of γ.

Size of a composition $|\gamma| = c_1 + c_2 + \ldots + c_k$

$k = 0 : ()$

$k = 1 : (1)$

$k = 2 : (2),\ (1,\ 1)$

$k = 3 : (3),\ (2,\ 1),\ (1,\ 2),\ (1,\ 1,\ 1)$

Let $\mathcal{C}$ be the set of all compositions.
Consider the power series $C(x) = \sum_{\gamma \in \mathcal{C}} x^{|\gamma|} = \sum_{n=0}^{\infty} C_n x^n$
where C_n is the number of compositions of size n.

No parts: only $()$. there is one
One part: (c). for some $c \in \mathbb{P}$

$$\sum_{c=1}^{\infty} x^c = x + x^2 + \ldots = \frac{x}{1-x}$$

Two parts: (a, b). for some $a, b \in \mathbb{P}$.

$$\sum_{(a,b) \in \mathbb{P}^2} x^c = \sum_{a=1}^{\infty} x^a \sum_{b=1}^{\infty} x^b = (\frac{x}{1-x})^2$$

k parts: $(c_1, c_2, \ldots, c_k)$

$$\sum_{(c_1, c_2, \ldots, c_k) \in \mathbb{P}^k} x^{c_1 + \ldots + c_n} = (\frac{x}{1-x})^k$$

Any number of parts ($k \in \mathbb{N}$)

$$\begin{aligned} C(x) &= \sum_{k=0}^{\infty} (\frac{x}{1-x})^k \\ &= \frac{1}{1 - \frac{x}{1-x}} \\ &= \frac{1-x}{1-2x} \\ &= 1 + \frac{x}{1-2x} \\ &= 1 + \sum_{n=1}^{\infty} 2^{n-1} x^n \end{aligned}$$

So $C_n = \begin{cases} 1 & \text{if } n = 0 \\ 2^{n-1} & \text{if } n \geq 1 \end{cases}$

Check 8 compositions of size 4

2.1 Compositions with restrictions

eg All parts are at least 2

No parts: () $1 = (\frac{x^2}{1-x})^0$

One part: (c) with $c \geq 2$ $\sum_{c=2}^{\infty} x^c = \frac{x^2}{1-x}$

Two parts: lazy to type it...

k parts: $(c_1, \ldots, c_k)$ each $c_i \geq 2$ $(\frac{x^2}{1-x})^k$

Any number of parts:

$$\sum_{k=0}^{\infty} (\frac{x^2}{1-x})^k = \ldots = 1 + \frac{x^2}{1-x-x^2}$$

$1, 0, 1, 1, 2, 3, 5, 8, 13, \ldots$ (Fibonacci Numbers)

eg All parts are at most 2.

No parts: () $1 = (x + x^2)^0$

One part: $(1), (2)$ $\sum_{c=2}^{\infty} x^c = x + x^2$

$\vdots$

k parts: $(c_1, \ldots, c_k)$ each $c_i \in \{1, 2\}$ $(x + x^2)^k$

Any number of parts:

$$\sum_{k=0}^{\infty} (x + x^2)^k = \ldots = 1 + \frac{1}{1-x-x^2}$$

$1, 1, 2, 3, 5, 8, 13, \ldots$ (Fibonacci Numbers)

It follows that for all $n \in \mathbb{N}$:
the number of compositions with all parts ≤ 2 of size n

equals

the number of compositions with all parts ≥ 2 of size $(n+2)$

eg

parts ≤ 2 $(n = 5)$		parts ≥ 2 $(n + 2 = 7)$
(2,2,1)		(7)
(2,1,2)		(5,2)
(1,2,2)		(2,5)
(2,1,1,1)	$\longleftrightarrow$	(4,3)
(1,2,1,1)		(3,4)
(1,1,2,1)		(3,2,2)
(1,1,1,2)		(2,3,2)
(1,1,1,1,1)		(2,2,3)

Eg: Compositions in which part is odd

- Single part is in $\{1,3,5,\dots\} = \mathbb{O}$
- k parts $(c_1, c_2, \dots, c_k) \in \mathbb{O}^k$
- Number of parts("length") is any $k \in \mathbb{N}$

---

* one part: $x + x^3 + x^5 + \dots = \sum_{i=0}^{\infty} x^{2i+1} = x \sum_{i=0}^{\infty} (x^2)^i$

* k parts: $(\frac{x}{1-x^2})^k$

* arbitrary $k \in \mathbb{N}$

$$\sum_{k=0}^{\infty} (\frac{x}{1-x^2})^k = \dots = 1 + \frac{x}{1-x-x^2}$$

(More Fibonacci Numbers)

Eg

- Every part is at most 3
- The number of parts is even

---

* One part: $c \in \{1,2,3\}$ $\quad \sum_{c=1}^{3} x^c = x + x^2 + x^3$

* k parts $(x + x^2 + x^3)^k$

* $k = 2j$ is even

$$\sum_{j=0}^{\infty} (x + x^2 + x^3)^{2j} = \frac{1}{1-(x+x^2+x^3)^2}$$

2.2 Theory of Generating Functions

In the coursenote, it is in §3.

Given a set Ω of "objects".
Given a weight function: $\quad \omega : \Omega \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, the set $\omega^{-1}(n) = \{\sigma \in \Omega : \omega(\sigma) = n\}$ is finite

The generating function of Ω with respect to ω is

$$\Phi_{\Omega}^{\omega}(x) = \sum_{\sigma \in \Omega} x^{\omega(\sigma)}$$

- get a formula for $\Phi_{\Omega}^{\omega}(x)$
- use that to extract the coefficient of $\Phi_{\Omega}^{\omega}(x)$

So $\Phi_{\Omega}^{\omega}(x) = \sum_{n=0}^{\infty} c_n x^n$

Let $c_n = \underbrace{[x^n]\Phi_\Omega^\omega(x)}_{\substack{\text{coefficient of } x^n \text{ is} \\ \Phi_\Omega^\omega(x)}}$

Then

$$\begin{aligned}
\Phi_\Omega^\omega(x) &= \sum_{\sigma\in\Omega} x^{\omega(\sigma)} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{\substack{\sigma\in\Omega: \\ \omega(\sigma)=n}} 1\Big)x^n \\
&= \sum_{n=0}^{\infty} |\omega^{-1}(n)|x^n \\
&= \sum_{n=0}^{\infty} c_n x^n
\end{aligned}$$

So $[x^n]\Phi_\Omega^\omega(x)$ is the number of objects in Ω of weight $n \in \mathbb{N}$

<u>Sum Lemma:</u> If $A = B \bigcup C$ is a disjoint union of sets ($B \bigcap C = \varnothing$), then $|A| = |B| + |C|$ (If A is finite)
If $\omega : A \to \mathbb{N}$ is a weight function, then (by restriction) it is also a weight function on B and on C.
Now

$$\begin{aligned}
\Phi_A^\omega(x) &= \sum_{\sigma\in A} x^{\omega(\sigma)} \\
&= \sum_{\sigma\in B\bigcup C} x^{\omega(\sigma)} \\
&= \sum_{\sigma\in B} x^{\omega(\sigma)} + \sum_{\sigma\in C} x^{\omega(\sigma)} \\
&= \Phi_B^\omega(x) + \Phi_C^\omega(x)
\end{aligned}$$

<u>Product Lemma:</u> Let A, B be the sets with weight functions: $\begin{cases} \alpha : A \to \mathbb{N} \\ \beta : B \to \mathbb{N} \end{cases}$

Consider the set $A \times B$ with the function $\omega(r, s) = \alpha(r) + \beta(s)$
Then $\omega : A \times B \to \mathbb{N}$ is a weight function, and

$$\begin{aligned}
\Phi_{A\times B}^\omega(x) &= \sum_{(r,s)\in A\times B} x^{\alpha(r)+\beta(s)} \\
&= \sum_{r\in A} x^{\alpha(r)} \cdot \sum_{s\in B} x^{\beta(s)} \\
&= \Phi_A^\alpha(x) \cdot \Phi_B^\beta(x)
\end{aligned}$$

Infinite Sum Lemma
Let $A = \bigcup_{k=0}^{\infty} A_k$ be a disjoint union, and $\omega : A \to \mathbb{N}$ a weight function. Then

$$\Phi_A^\omega(x) = \sum_{\sigma\in A} x^{\omega(\sigma)} = \sum_{k=0}^{\infty}\sum_{\sigma\in A_k} x^{\omega(\sigma)} = \sum_{k=0}^{\infty} \Phi_{A_k}^\omega(x)$$

Finite String Lemma
For a set S, let $S^* = \bigcup_{k=0}^{\infty} S^k$ where $S^k = \underbrace{S \times S \times \ldots \times S}_{k \text{ factors}}$
Let $\omega : S \to \mathbb{N}$ be a weight function.
Define : $\omega^* : S^* \to N$ by $\omega^*(\sigma_1, \sigma_2, \ldots, \sigma_k) = \omega(\sigma_1) + \ldots + \omega(\sigma_k)$

($*$) Condition

If S has no elements of weight zero, then ω^* is a weight function on S^*.
If $\omega(\alpha) = 0$ for some $\alpha \in S$, then $(\alpha, \dots, \alpha) \in S^k$ has $\omega^*(\alpha, \dots, \alpha) = 0$.
So $(\omega^*)^{-1}(0)$ is infinite.

$\Phi^{\omega^*}_{S^k}(x) = \left(\Phi^{\omega}_{S}(x)\right)^k$ by the Product Lemma

By thee infinite sum Lemma

$$\Phi^{\omega^*}_{S^k}(x) = \sum_{k=0}^{\infty} \left(\Phi^{\omega}_{S}(x)\right)^k = \frac{1}{1 - \Phi^{\omega}_{S}(x)}$$

eg. (violate the condition)
If $S = \mathbb{N} = \{0, 1, 2, \dots\}$ with $\omega(i) = i$

$$\Phi_{\mathbb{N}}(x) = x^0 + x^1 + \dots = \frac{1}{1-x}$$

So $\dfrac{1}{1 - \Phi_{\mathbb{N}}(x)} = \dots = -\dfrac{1}{x} + 1$

2.2.1 Compositions Revisited

$\gamma = (c_1, c_2, \dots, c_k)$ each $c_i \in \mathbb{P} = \{1, 2, \dots\}$. $k \in \mathbb{N}$. $|\gamma| = c_1 + \dots + c_k$
Now $\Phi_{\mathbb{P}}(x) = x + x^2 + x^3 + \dots = \dfrac{x}{1-x}$. Set of all compositions is $\mathbb{P}^*$

By the String Lemma $\Phi_{\mathbb{P}^*}(x) = \dfrac{1}{1 - (\frac{x}{1-x})} = 1 + \dfrac{x}{1-2x}$

$$\text{Number of compositions of size } n = \begin{cases} 1 & \text{if } n = 0 \\ 2^{n-1} & \text{if } n \geq 1 \end{cases}$$

P: subsets of $\{1, 2, \dots, n\}$
$\mathcal{C}_{n+1}$: compositions of size $n+1$

$\mathcal{P}_n \leftrightarrows \mathcal{C}_{n+1}$ Then $|\mathcal{P}_n| = |\mathcal{C}_{n+1}| = 2^n$

Sort $S : s_1 < s_2 < \dots < s_k$

$$\begin{array}{ccc} S = \{s_1, s_2, \dots, s_l\} & \longleftrightarrow & \gamma = (c_1, c_2, \dots, c_k) \\ |S| = l & & |\gamma| = c_1 + \dots + c_k = n+1 \end{array}$$

$S \mapsto$ put $k = l + 1$ and $\gamma = (s_1, s_2 - s_1, s_3 - s_2, \dots, (n+1) - s_l)$ (with $s_0 = 0$)
put $S = \{c_1, c_1 + c_2, c_1 + c_2 + c_3, \dots, \sum_{i=1}^{k-1} c_i\} \mapsfrom \gamma = (c_1, c_2, \dots, c_k)$

Eg $n = 2$

$\mathcal{P}_2$	$\mathcal{C}_3$
$\{\}$	(3)
$\{1\}$	$(1, 2)$
$\{2\}$	$(2, 1)$
$\{1, 2\}$	$(1, 1, 1)$

eg $n = 10$ and $S = \{2, 3, 5, 7\} \leftrightarrow (2, 1, 2, 2, 4)$

eg Subsets of $\{1, 2, \dots, n\}$ that have no "gaps" of size 2.

eg $n = 11$. $S = \{2,3,6,9\}_{\text{ in } \mathcal{P}_{11}} \leftrightarrow (2, \underbrace{1, \quad 3, \quad 3}_{\text{gaps can't be 2}}, 3)$ in $\mathcal{C}_{12}$

In composition, first & last can be any positive integers

- Let $\mathcal{A}_n$ be subsets of n with no gaps of size 2
- Let $\mathcal{B}_{n+1}$ be compositions of size $n+1$ meeting the condition above

Then $|\mathcal{A}_n| = |\mathcal{B}_{n+1}|$

$\mathbb{P} = \{1,2,3,\dots\}$ $\quad \Phi_{\mathbb{P}}(x) = \frac{x}{1-x}$

$\mathbb{G} = \{1, \quad 3,4,\dots\}$ $\quad \Phi_{\mathbb{G}}(x) = x + \frac{x^3}{1-x}$ $\qquad$ Note: $\mathbb{G}$ is ($\mathbb{P}$ missing 2)

	k : number of parts of γ		
0	$\gamma = ()$	$\{()\}$	1
1	(c_1)	$\mathbb{P}$	$\frac{x}{1-x}$
2	(c_1, c_2)	$\mathbb{P} \times \mathbb{P}$	$(\frac{x}{1-x})^2$
3	(c_1, c_2, c_3)	$\mathbb{P} \times \mathbb{G} \times \mathbb{P}$	
$\vdots$			
k for $k \geq 2$	$(c_1, \dots, c_k)$	$\mathbb{P} \times \mathbb{G}^{k-2}\mathbb{P}$	$(\frac{x}{1-x})^2(x + \frac{x^3}{1-x})^{k-2}$

By the infinite sum Lemma, the generating function for the compositions is

$$\begin{aligned} &1 + \frac{x}{1-x} + \sum_{k=2}^{\infty}(\frac{x}{1-x})^2(x + \frac{x^3}{1-x})^{k-2} \\ &= \frac{1}{1-x} + (\frac{x}{1-x})^2 \frac{1}{1 - \frac{x-x^2+x^3}{1-x}} \\ &= \frac{1-2x+2x^2-x^3}{(1-x)(1-2x+x^2-x^3)} \end{aligned}$$

Question For $n \in \mathbb{N}$, among all compositions of size n, what is the average number of parts of size one?

		avg
$n = 0$	$()$	0
$n = 1$	(1)	1
$n = 2$	(2), $(1,1)$	1
$n = 3$	(3), $(2,1)$, $(1,2)$, $(1,1,1)$	$\frac{5}{4}$
$n = 4$	...	$\frac{?}{8}$

In general: the formula is

$$\frac{\text{total numbers of parts of size 1 (among compositions of size } n)}{\text{number of compositions of size } n}$$

For $n \geq 1$, the denominator is 2^{n-1}

Set of all compositions is $\mathcal{C} = \mathbb{P}^*$ where $\mathbb{P} = \{1,2,3,\dots\}$

$x^{|\gamma|}$ and $y^{m_1(\gamma)}$ for a composition $\gamma = (c_1, c_2, \dots, c_k) \in \mathbb{P}^k$ for some $k \in \mathbb{N}$ where $|\gamma| = c_1 + c_2 + \dots + c_k$ and $m_1(\gamma) = |\{i : 1 \leq i \leq k(\gamma) \text{ and } c_i = 1\}|$

Let $C(x,y) = \sum_{\gamma\in\mathcal{C}} x^{|\gamma|} y^{m_1(\gamma)}$

Consider

$$\begin{aligned}\frac{\partial}{\partial y} C(x,y)\Big|_{y=1} &= \frac{\partial}{\partial y} \sum_{\gamma\in\mathcal{C}} x^{|\gamma|} y^{m_1(\gamma)}\Bigg|_{y=1} \\ &= \sum_{n=0}^{\infty} (\sum_{\substack{\gamma\in\mathcal{C}\\|\gamma|=n}} m_1(\gamma)) x^n\end{aligned}$$

<u>One part</u>: (c) with $c \in \mathbb{P}$

$$xy + x^2 + x^3 + x^4 + \ldots = xy + \frac{x^2}{1-x} = \frac{x}{1-x} + (y-1)x$$

<u>k parts</u> $(xy + \frac{x^2}{1-x})^k$

<u>Any parts</u>

$$\sum_{k=0}^{\infty} (xy + \frac{x^2}{1-x})^k = \frac{1}{1-(xy+\frac{x^2}{1-x})} = \frac{1-x}{(1-x) - xy(1-x) - x^2} = \frac{1-x}{1-x-x^2+y(x^2-x)}$$

$$\begin{aligned}\frac{\partial}{\partial y} C(x,y)\Big|_{y=1} &= (1-x)\frac{1(-1)(x^2-x)}{(1-x-x^2+1(x^2-x))^2} \\ &= \frac{x(1-x)^2}{(1-2x)^2} \\ &= (x-2x^2+x^3)\sum_{j=0}^{\infty}(j+1)2^j x^j \\ &= x + (4-2)x^2 + \sum_{n=3}^{\infty}(n\cdot 2^{n-1} - 2(n-1)2^{n-2} + (n-2)2^{n-3})x^n\end{aligned}$$

Total number of parts of size 1 among compositions of size n

n	
0	0
1	1
2	2
$n \geq 3$	$2^{n-3}(4n-4(n-1)+n-2)$ $= (n+2)2^{n-3}$

Average number of parts of size 1 among all compositions of size $n \geq 3$ is

$$\frac{2^{n-3}(n+2)}{2^{n-1}} = \frac{n+2}{4}$$

3 Binary Strings

It's in §5

$\{0,1\}^* = \bigcup_{k=0}^{\infty} \{0,1\}^k$ For $\sigma \in \{0,1\}^*$, k is the length $l(\sigma)$. 2^k binary strings of length k.

$$\sum_{\sigma \in \{0,1\}^*} x^{l(\sigma)} = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$$

k	strings
0	ε means empty string
1	0, 1
2	00, 01, 10, 11
...	...

How many binary strings of length k have no consecutive 0 s?

k	number
0	1
1	2
2	3
3	5
4	8
...	...

Wow... it's Fibonacci numbers, we will prove it later.

3.1 Rational Languages

$\mathcal{L} \subseteq \{0,1\}^*$ a subset of binary strings that can be described by a Regular Expression (RE).

3.1.1 Regular Expression

- $\varepsilon, 0, 1$ are REs.
- If R and S are REs, the $\mathsf{R} \smile \mathsf{S}, \mathsf{RS}$ are REs.
- If R is a RE and $k \in \mathbb{N} = \{0, 1, \ldots\}$, then R^k is a RE.
- If R is a RE, then R^* is a RE.

A RE R produces a rational language $\mathcal{R} : \mathsf{R} \rhd \mathcal{R}$ (recursively)

3.1.2 Concatenation Product

For $\alpha = a_1 a_2 \ldots a_k$ and $\beta = b_1 b_2 \ldots b_l$ in $\{0,1\}^*$, their concatenation is

$$\alpha\beta = a_1 \ldots a_k b_1 \ldots b_l$$

For sets $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^*$, the concatenation product is

$$\mathcal{A}\mathcal{B} = \{\alpha\beta : \alpha \in \mathcal{A} \text{ and } \beta \in \mathcal{B}\}$$

EG: $\mathcal{A} = \{011, 01\}$ $\mathcal{B} = \{100, 1100\}$

$$\mathcal{AB} = \{011100, 0111100, 01100, 011100\}$$

Notice that the first and last element are same, which means same string produced twice. Thus $|\mathcal{AB}| = 3$

RE: $\mathsf{R} \rhd \mathcal{R}$ produces a rational language $\mathcal{R}$.

(Assume $\mathsf{R} \rhd \mathcal{R}$ and $\mathsf{S} \rhd \mathcal{S}$)

- $\varnothing$ produces $\varnothing$
- $\varepsilon, 0$ and 1 produce $\{\varepsilon\}, \{0\}, \{1\}$ respectively.
- $\mathsf{R} \smile \mathsf{S}$ produces $\mathcal{R} \bigcup \mathcal{S}$
- RS produces $\mathcal{RS}$
- R^k produces $\mathcal{R}^k$
- R^* produces $\mathcal{R}^*$

For any $\mathcal{R} \subseteq \{0,1\} : \mathcal{R}^0 = \{\varepsilon\}$

Examples

- $(0 \smile 1)^*$ produces $\{0,1\}^*$, all binary strings (exactly one each)
- $(0 \smile 1 \smile 10)^*$ produces all binary strings (sometimes more than once)

$$0.1.1.(10).0.1.0.1 = 0.1.1.1.0.0.(10).1$$

- (recall last lecture) Binary strings with no two consecutive 0s
  Consider 011101101011101110

  0,1,1,10,1,10,10,1,1,10,1,1,10 read §5 to have a better understanding

$$(\varepsilon \smile 0)(1 \smile 10)^*$$

produces them exactly once each

REs <u>lead to</u> rational functions $\mathsf{R} \rightsquigarrow R(x)$

Assume $\mathsf{R} \rightsquigarrow R(x)$ $\mathsf{S} \rightsquigarrow S(x)$

- $\varepsilon, 0, 1$ lead to $1, x, x$ respectively
- $\mathsf{R} \smile \mathsf{S} \rightsquigarrow R(x) + S(x)$
- $\mathsf{RS} \rightsquigarrow R(x)S(x)$
- $\mathsf{R}^k \rightsquigarrow R(x)^k$
- $\mathsf{R}^* \rightsquigarrow \dfrac{1}{1 - R(x)}$

Ex:

- $(0 \smile 1)^* \rightsquigarrow \dfrac{1}{1-(x+x)} = \dfrac{1}{1-2x}$ good!
- $(1 \smile 1 \smile 10)^* \rightsquigarrow \dfrac{1}{1-(x+x+x^2)} = \dfrac{1}{1-2x-x^2}$ bad!
- $(\varepsilon \smile 0)(1 \smile 10)^* \rightsquigarrow (1+x)\dfrac{1}{1-(x+x^2)} = \dfrac{1+x}{1-x-x^2}$ good ~~or bad~~

Next: How to verify that it produces exactly once?

Read HW #2 B2 see question 5.14(a)

3.2 Unambiguous Expressions

Definition A RE R that produces a rational language $\mathsf{R} \triangleright \mathcal{R}$ and every string in $\mathcal{R}$ is produced exactly once.

What makes a RE unambiguous?

- $\varepsilon, 0$, and 1 are unambiguous
- $\mathsf{R} \smile \mathsf{S}$ is unambiguous if and only if $\mathcal{R} \bigcap \mathcal{S} = \varnothing$. $\mathcal{R}$ and $\mathcal{S}$ are disjoint
- RS is unambiguous if and only if
  there is a bijection $\mathcal{RS} \rightleftharpoons \mathcal{R} \times \mathcal{S}$
  I.e. Every string $\sigma \in \mathcal{RS}$ is produced exactly once as a concatenation $\sigma = \alpha\beta$ with $\alpha \in \mathcal{R}$ and $\beta\mathcal{S}$
- R^k is unambiguous if and only if

$$\underbrace{\mathcal{R}^k}_{\text{concatenation}} \rightleftharpoons \underbrace{\mathcal{R} \times \ldots \times \mathcal{R}}_{k \text{ factors (cartesian)}}$$

- R^* is unambiguous if and only if, in the expression $\mathcal{R}^* = \bigcup\limits_{k=0}^{\infty} \mathcal{R}^k$, each $\mathcal{R}^k$ is unambiguous and the union is disjoint.

eg $0^*(1^*10^*0)^*1^*$

Unambiguous or not? What RL does it produce?

$$\varepsilon.(110)(10)(11000)(111100)(10)(10)(11)$$

Unambiguous, produce $\{0,1\}^*$
This leads to $\rightsquigarrow \dfrac{1}{1-x} \cdot \dfrac{1}{1-\dfrac{x}{1-x}\cdot\dfrac{x}{1-x}} \cdot \dfrac{1}{1-x} = \dfrac{1}{1-2x}$
produces every string in $\{0,1\}^*$ block by block

eg $1^*(0^*01^*1)^*0^*$ similarly

3.2.1 Blocks

Definition A <u>block</u> in a string $\sigma = b_1 b_2 \ldots b_l \in \{0,1\}^*$ is a non-empty <u>maximal</u> subsequence of consecutive equal bits.

$$\underbrace{11}\,\underbrace{0}\,\underbrace{000}\,\underbrace{111}\,\underbrace{000}\,\underbrace{1}\,\underbrace{00}$$

eg Strings with no two consecutive 0s

$$\left.\begin{array}{ll} A = (\varepsilon \smile 0)(1^*10)^*1^* & \text{block decomposition} \\ B = (\varepsilon \smile 0)(1 \smile 10)^* & \text{not a block decomposition} \end{array}\right\}\text{both unambiguous}$$

$$A \rightsquigarrow (1+x)\cdot\frac{1}{1-\frac{x^2}{1-x}}\cdot\frac{1}{1-x} = \frac{1+x}{1-x-x^2}$$

$$B \rightsquigarrow (1+x)\cdot\frac{1}{1-(x+x^2)} = \frac{1+x}{1-x-x^2}$$

eg Strings that don't contain 110 as a substring.
Block decomposition

$$0^*(10^*0)^*1^* \rightsquigarrow \frac{1}{1-x}\cdot\frac{1}{1-\dfrac{x^2}{1-x}}\cdot\frac{1}{1-x} = \frac{1}{1-2x+x^3}$$

Example Among all 2^n binary strings of length n, what is the average number of blocks?
$\sigma = \{0,1\}^*$
$\ell(\sigma) = \text{length of } \sigma$
$b(\sigma) = \text{number of blocks of } \sigma$

$$B(x,y) = \sum_{\sigma\in\{0,1\}^*} x^{\ell(\sigma)}y^{b(\sigma)}$$

One block (Either $0*0$ or $1*1$)

$$(x+x^2+x^3+x^4+\cdots)\cdot y^1 = \frac{xy}{1-x}$$

1 block

Block decomposition of $\{0,1\}^*$

$$(\varepsilon \smile 0^*0)(1^*10^*0)^*(\varepsilon \smile 1^*1)$$

$$\rightsquigarrow \left(1+\frac{xy}{1-x}\right)^2\frac{1}{1-(\frac{xy}{1-x})^2} = \frac{(1-x+xy)^2}{(1-x-xy)(1-x+xy)} = \frac{1-x+xy}{1-x-xy} = B(x,y)$$

the total number of blocks among all binary strings of length n is coefficient of x^n is ...

$$\begin{aligned}
[x]\frac{\partial}{\partial y}B(x,y)\Big|_{y=1} &= \left[\frac{x}{1-x-xy} + \frac{(1-x+xy)(-1)(-x)}{(1-x-xy)^2}\right]\Bigg|_{y=1} \\
&= \frac{2(x-x^2)}{(1-2x)^2} \\
&= 2x\sum_{i=0}^{\infty}\binom{i+1}{1}2^i - 2x^2\sum_{j=0}^{\infty}\binom{j+1}{1}2^jx^j \\
&= 2x + \sum_{n=2}^{\infty}(n\cdot 2^n - (n-1)\cdot 2^{n-1})x^n
\end{aligned}$$

The total number of blocks among all $\sigma \in \{0,1\}^*$ of length n is $\begin{cases} 0 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ (n+1)2^{n-1} & \text{if } n \geq 2 \end{cases}$

The average number of blocks per string among all $\sigma \in \{0,1\}^n$ is $\begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \frac{n+1}{2} & \text{if } n \geq 2 \end{cases}$

eg $n = 3$

σ	$b(\sigma)$
000	1
001	2
010	3
100	2
110	2
101	3
011	2
111	1

$\sum b(\sigma) = 16$
average number = 16/8 = 2

3.2.2 Another Example

Example Strings in the set $\{a, b, c\}^*$

(blocks) $\underbrace{bb}|@|\underbrace{b|cc|b}|@|\underbrace{cc|b}|@|\underbrace{cc}|aaa|\underbrace{bb|c}|@|\underbrace{cc|b}$

Block decomposition between blocks of a_s are nonempty. Strings in $\{b, c\}^*$.
Let $\mathcal{G}$ [1] be a block decomposition for nonempty strings in $\{b, c\}^* \setminus \{\varepsilon\}$

Then $a^*(\mathcal{G}a^*a)^*(\varepsilon \smile \mathcal{G})$ is a block decomposition for $\{a, b, c\}^*$

3.3 Ternary Strings

Ternary strings $\{a, b, c\}^*$
Block decomposition for $\{a, b, c\}^*$: G : block decomposition for nonempty strings of $\{b, c\}^*$

$$a^*(Ga^*a)^*(\varepsilon \smile G) \qquad \text{good}$$

$$\begin{aligned} G &= b^*b(c^*cb^*b)^*c^* \smile c^*c(b^*bc^*c)^*b^* \\ &= b^*(c^*cb^*b)^*c^* \setminus \varepsilon \end{aligned}$$

3.3.1 Block patterns in $\{a, b, c\}^*$

String: $\underbrace{bb}_{B}\,\underbrace{a}_{A}\,\underbrace{ccc}_{C}\,\underbrace{aa}_{A}\,\underbrace{bbb}_{B}\,\underbrace{cc}_{C}\,\underbrace{bb}_{B}\,\underbrace{cc}_{C}\,\underbrace{aa}_{A}\,\underbrace{b}_{B}\,\underbrace{c}_{C}\,\underbrace{a}_{A}\,\underbrace{bb}_{B}\,\underbrace{cc}_{C}\,\underbrace{a}_{A}\,\underbrace{b}_{B}\,\underbrace{cc}_{C}$

Block patterns: $BACABCBCABCABCABC$
in $\{A, B, C\}^*$ with no consecutive letters. Call this set $\mathcal{D}$

Generating function for $\mathcal{D}$ keeping track of number of letters A, B, C separately.
Set of all strings $\{a, b, c\}^*$

$$\frac{1}{1 - (a + b + c)} = \sum_{k=0}^{\infty} (a + b + c)^k$$

$$\Phi_{\mathcal{D}}(A, B, C) = \sum_{\substack{\text{blocks} \\ \text{patterns} \\ \text{for } \{a,b,c\}^*}} A^{\#\text{ of } A_s} B^{\#\text{ of } B_s} C^{\#\text{ of } C_s}$$

Block of a_s $A = \{a, aa, aaa, \dots\} = a^*a$
Generating functions: $A = \frac{a}{1-a}$ $B = \frac{b}{1-b}$ $C = \frac{c}{1-c}$

[1]This is `\euler{G}`. In the coursenote, this font appears to be `\mathcal{G}` in `eulervm` package. When I try to use this package, it rendered my whole document to this font... Too bad! So I gave up trying...
[1] 我花了好久终于让这个脚注好使了。
[1] Emm... 我在教授个人网站上找到了解决方法了...

Question What is $\Phi_{\mathcal{D}}(\frac{a}{1-a}, \frac{b}{1-b}, \frac{c}{1-c})$?
It is $\frac{1}{1-(a+b+c)}$. since every string in $\{a,b,c\}^*$ is produced exactly once.

Now, from
$$A = \frac{a}{1-a} \implies A - aA = a \implies a = \frac{A}{1+A}$$
Similarly, $b = \frac{B}{1+B}$ and $c = \frac{B}{1+B}$
So
$$\Phi_{\mathcal{D}}(A,B,C) = \left(1 - \left(\frac{A}{1+A} + \frac{B}{1+B} + \frac{C}{1+C}\right)\right)^{-1}$$

Eg $\mathcal{L} \subset \{a,b,c\}^*$

- blocks of a_s have even length
- blocks of s_s have the length ≤ 2
- blocks of c_s have no restriction

$A = \{aa, aaaa, \dots\} = (aa)^* aa \qquad A = \dfrac{a^2}{1-a^2} \qquad$ So $\dfrac{A}{1+A} = \dfrac{a^2/(1-a^2)}{1/(1-a^2)} = a^2$

$B = \{b, bb\} \qquad$ So $\dfrac{B}{1+B} = \dfrac{b+b^2}{1+b+b^2}$

$C = c^* c \qquad$ so $\dfrac{C}{1+C} = \dfrac{c/(1-c)}{1/(1-c)} = c$
$$\Phi_{\mathcal{L}}(a,b,c) = \Phi_{\mathcal{D}}(A,B,C) = \left(1 - \left(a^2 + \frac{b+b^2}{1+b+b^2} + c\right)\right)^{-1}$$
$\Phi_{\mathcal{L}}(x,x,x)$ enumerates $\mathcal{L}$ with respect to length.

3.4 Transition Diagrams (Finite State Machines)

Example $\mathcal{L} \subseteq \{a,b,c\}^*$ consisting of strings that do not contain ac, bbb, or cba as consecutive substrings.
Inductive construction ... one letter at a time. Please see Figure 3.1.
Note: black circle nodes are **States**. The green arrows are **Transitions**.

3.4.1 Transition Matrix M

Square, indexed by states
$$M_{ij} = \begin{cases} 1 & \text{if there is a transition from stat } j \text{ to state } i \\ 0 & \text{otherwise} \end{cases}$$
$$M = \begin{array}{c} \begin{array}{cccccc} \varepsilon & a & b & c & bb & cb \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} & \begin{array}{c} \varepsilon \\ a \\ b \\ c \\ bb \\ cb \end{array} \end{array}$$

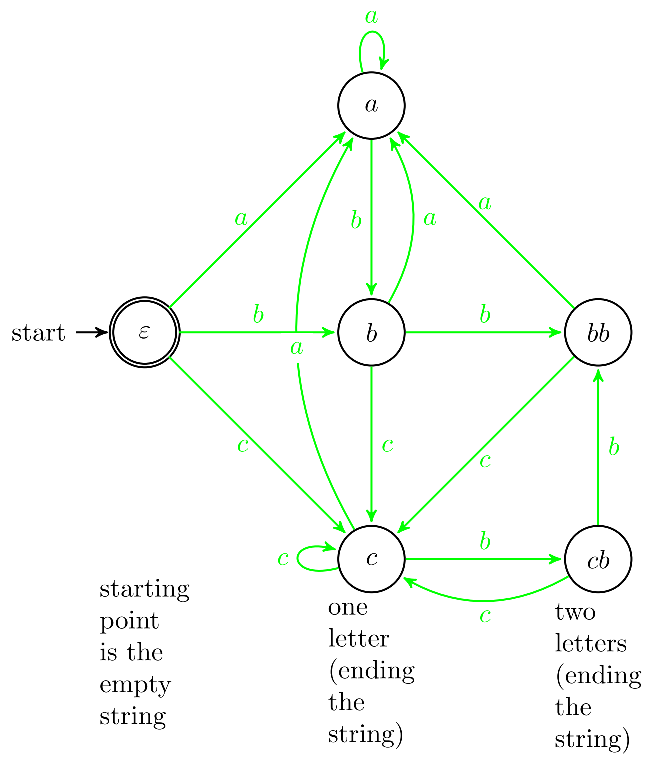


Figure 3.1: Our inductive construction

Let p_n be the vector in $\mathbb{Z}^6$ in which $(p_n)_j$ is the number of strings of length n in $\mathcal{L}$ in state j.

Then

$$Mp_n = p_{n+1}$$

Example

$$p_0 = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \quad \longrightarrow \quad Mp_0 = \begin{bmatrix}0\\1\\1\\1\\0\\0\end{bmatrix} = p_1 \quad \longrightarrow \quad Mp_1 = \begin{bmatrix}0\\3\\1\\2\\1\\1\end{bmatrix} = p_2$$

Proof By induction on n. Exercise. □

Generating function for $\mathcal{L}$ by states: $L(x) = \sum_{n=0}^{\infty} p_n x^n$

$$xML(x) = \sum_{n=0}^{\infty} p_{n+1}x^{n+1} = L(x) - p_0$$

$$p_0 = L(x) - xML(x) = (I - xM)L(x)$$

It turns out that $I - xM$ is invertible, so

$$L(x) = (I - xM)^{-1}p_0$$

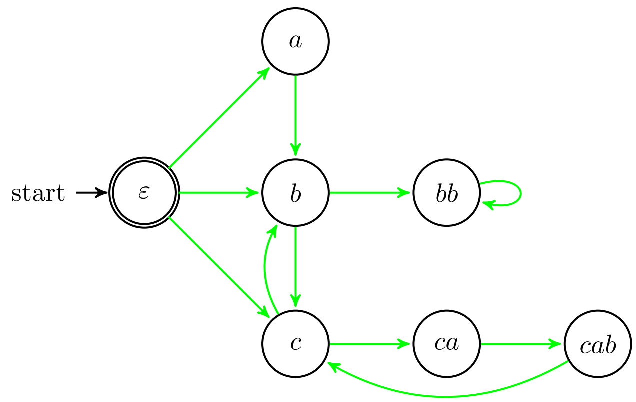


Figure 3.2: another example

Example String in $\{a, b, c\}^*$ call this set Ω See Figure 3.2
Forbidden substrings: $aa, ac, ba, bbc, cc, cabb$

States ε; single letters; proper prefixes of forbidden substrings

Transitions by appending a single letter

<u>**States:**</u> $\varepsilon; a, b, c; bb, ca, cab.$ $\quad M_{ij} = \begin{cases} 1 & \text{if } j \to i \\ 0 & \text{if not} \end{cases}$

$$M = \begin{array}{c} \\ \left(\begin{array}{ccccccc} \varepsilon & a & b & c & bb & ca & cab \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ \varepsilon \\ a \\ b \\ c \\ bb \\ cb \\ cab \end{array}$$

If $\vec{p}_n$ is the vector in $\mathbb{R}^7$, where $(\vec{p}_n)_j$ is the number of strings in Ω of length n in state j. Then $\vec{p}_{n+1} = M\vec{p}_n$ for $n \geq 0$. $\implies$ So $\vec{p}_n = M^n \vec{p}_0$ for all $n \geq 0$
So

$$Q(x) = \sum_{n=0}^{\infty} \vec{p}_n x^n = \sum_{n=0}^{\infty} (xM)^n \vec{p}_0 = (I - xM)^{-1} \vec{p}_0$$

<u>Note:</u> $\vec{p}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad Q(x) = \sum_{n=0}^{\infty} |\text{strings in } \Omega \text{ of length } n|\ x^n$

Let $\vec{v}_{\text{final}} = \vec{v}_{\text{fin}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

I am allowed to end with ε I am allowed to end with cab

Then $Q(x) = \vec{v}_{fin}(I - xM)^{-1}\vec{p}_0$

Example What about strings in Ω that (are non-empty and) end in b?

$\vec{v}_{fin} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

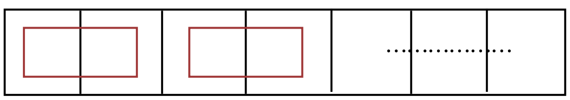
$$\vec{p}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \vec{p}_0 = \begin{bmatrix} 0 \\ x \\ x \\ x \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The "1" of first p_0 means x^0: start with ε but not end with it (non-empty). The "x" of second p_0 means x^1: start with a, b or c.

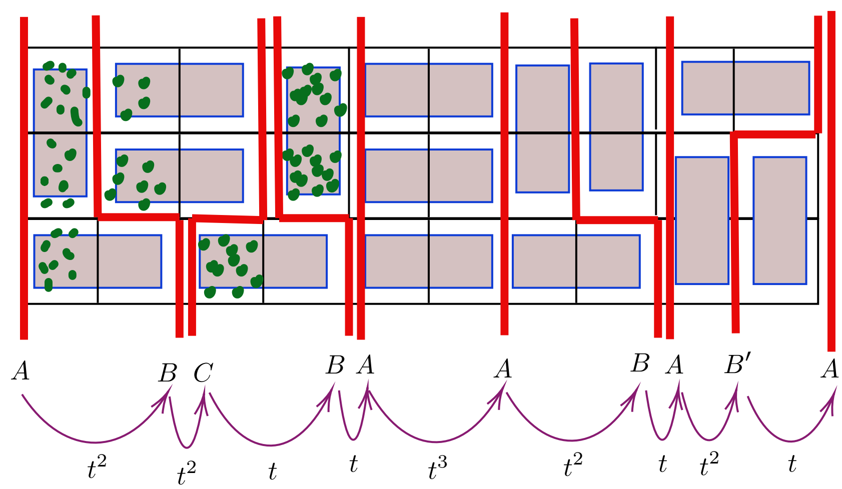
3.5 Domino Tilings

Fix $k \geq 1$. Tile a $k \times n$ rectangles with dominos (2×1 or 1×2 rectangles). How many ways to do it?

- $k = 1 : \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$



- $k = 2$ easy exercise. Fibonacci!
- $k = 3$ number of solutions is 0 if n is odd. Notation: $t^{\#\text{ of dominoes}}$

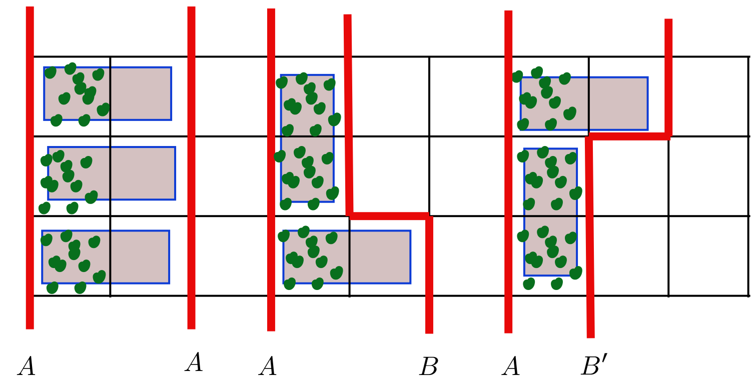


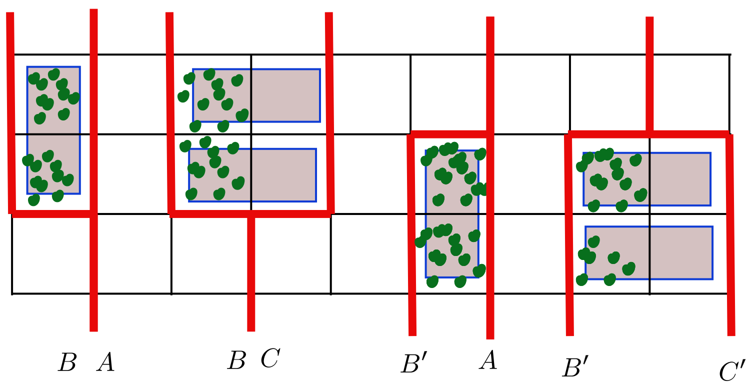
3.5.1 Generating function

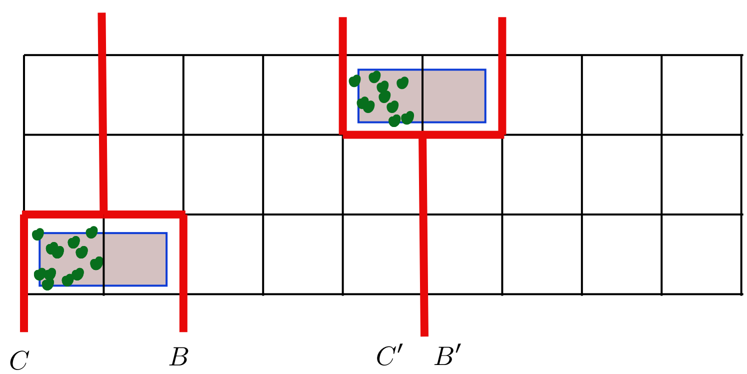
States: Shapes of the left-handed edge of the domino tiling

Transitions: Cover all squares in the left-most column to the right of the current state (green dots) with dominoes extending to the right (in all possible ways)

Weigh of the transition t^d if it uses d dominoes







We only need state A, B, B', C, C'. We can lump $B\&B'$ and $C\&C'$ by symmetry.
We only need states A, BB', CC'
Transitions & weights: $J \to m(t^d)I$ means state J goes to state I in m ways with weight t^d.

- $A \to (t^3)A, 2(t^2)BB'$
- $BB' \to (t^1)A, (t^2)CC'$
- $CC' \to (t^1)BB'$

Transition matrix T_{ij} is the sum of $m \cdot t^d$ over all transitions from state j to state i.

$$\begin{array}{cc} & \begin{array}{ccc} A & BB' & CC' \end{array} \\ T = & \begin{pmatrix} t^3 & t & 0 \\ 2t^2 & 0 & t \\ 0 & t^2 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ BB' \\ CC' \end{array}$$

Let $A = I - T$ and $A^{-1} = (I-T)^{-1} = \sum_{k=0}^{\infty} T^k$ has ij-entry $(A^{-1})_{ij} = \sum_{\substack{\text{domino tilings} \\ \text{left edge in state } j \\ \text{right edge in state } i}} t^{\#\text{ of dominoes}}$

We want the (A,A)-entry of the matrix A^{-1}.

$$A^{-1} = (I-T)^{-1} = \frac{1}{\det(I-T)} \operatorname{adj}(I-T)$$

We need to calculate $\det(I-T)$ and $\operatorname{adj}(I-T)$. (adj means adjoint)

$$A = I - T = \begin{bmatrix} 1-t^3 & -t & 0 \\ -2t^3 & 1 & -t \\ 0 & -t^2 & -1 \end{bmatrix}$$

$$\det A = 1 - 4t^3 + t^6$$

$$[\operatorname{adj}(I-T)]_{1,1} = \begin{vmatrix} 1 & -t \\ -t^2 & 1 \end{vmatrix} = 1 - t^3$$

So the generating function is $\dfrac{1-t^3}{1-4t^3+t^6}$. Let $t = x^{2/3}$, we get $A(x) = \sum_{n=0}^{\infty} a_n x^n = \dfrac{1-x^2}{1-4x^2+x^4}$. (Let a_n denote the number of domino tilings of a 3-by-n rectangle.)

4 Midterm Review

Reminder: Our midterm is on Tuesday, Nov.6, from 4:30 to 6:20 in room RCH 301. That is very close to our classroom.

- HW 1 2 3, but no Maple & one easy graph
- basic counting
  permutations
  subsets, multisets, cards, dice, ...
- Rational Functions & linear recurrence relation & Partial Fractions

$$\frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} g_n x^n$$

- Binomial Theorem & Binomial Series & Generating Functions
  Compositions & subsets with restriction
- Computing the average number of things in doohickey
- Binary Strings
  regular expression, unambiguous expression, block decomposition... etc
- Strings over larger alphabets

$$\frac{1}{1-(x+y+z)} = \left[1-\left(\frac{X}{1+X}+\frac{Y}{1+Y}+\frac{Z}{1+Z}\right)\right]^{-1} \qquad \text{with } X = \frac{x}{1-x}$$

  Proof not required.
- Transition diagrams & matrices

$$G(x) = v_{fin}(I-T)^{-1}v_{init} \quad T = xA$$

4.1 Omitted Topics

- Catalan Numbers
- Lattice Paths
- Stirling Strings
- More on CO 330

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5 Introduction

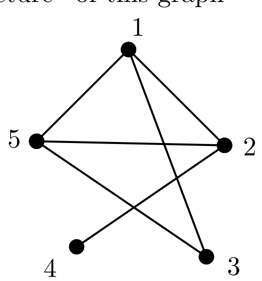
Graph Theory is an “abstract” theory of networks.

Two sets:

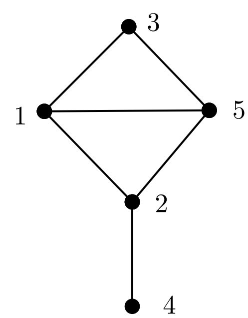
- V: a finite set of <u>vertices</u>
- E: a set of 2-element subsets of V, called <u>edges</u>

By convention, $|V| = p$, $\qquad |E| = q$
$G = (V, E)$ is a graph.

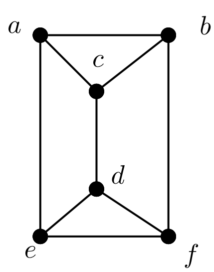
Example
$V = \{1, 2, 3, 4, 5\}$
$E = \{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$
“picture” of this graph



is the same as

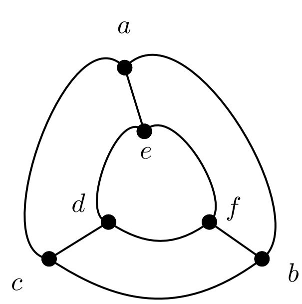


Example



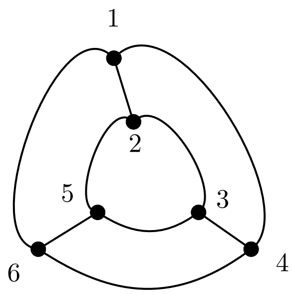
G

=



H

these represents the same graph. Since $V(G) = V(H)$, and $E(G) = E(H)$.



J

φ is an isomorphism from H to J.

v	$\varphi(v)$
a	1
b	4
c	6
d	5
e	2
f	3

We'd like to say that H and J are "the same" some how - changing "names" of the vertices but keeping the patterns of interconnection.

6 Graphs and Isomorphism

Definition An <u>isomorphism</u> from G to H is a function $f : V(G) \to V(H)$ such that

- f is a bijection, and
- for every $\{v, w\} \subseteq V(G)$:
  $\{f(v), f(w)\} \in E(H)$ if and only if $\{v, w\} \in E(G)$

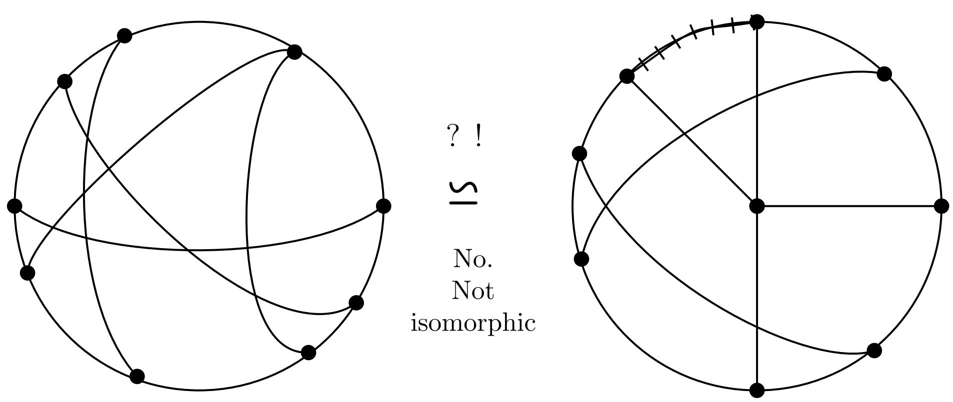
G and H are <u>isomorphic</u> if there is an isomorphism from G to H.

Denoted $G \simeq H$

Claim Isomorphism is an equivalence relation.

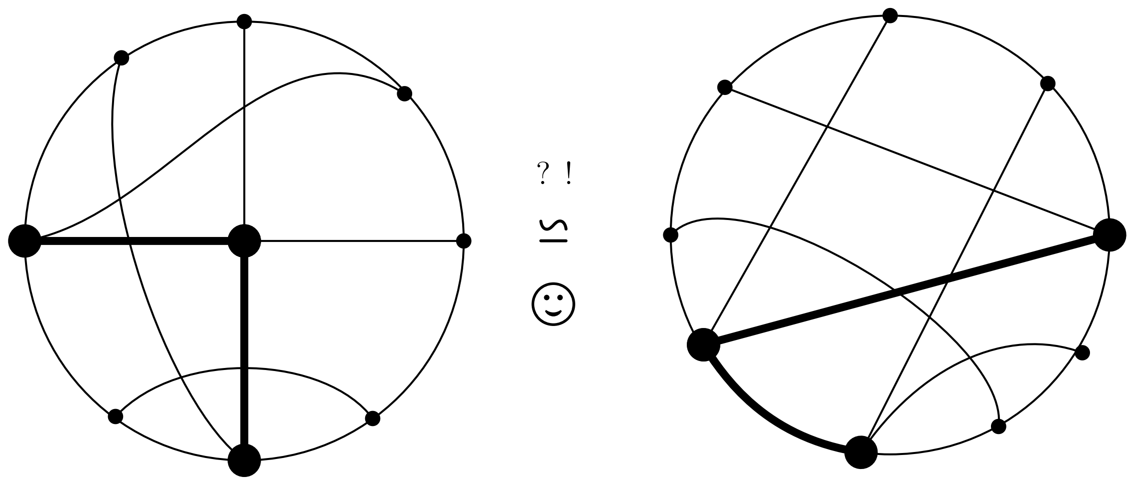
- reflexive: $G \simeq G$ (identity function)
- symmetric: if $G \simeq H$, then $H \simeq G$. (inverse)
- transitive: if $G \simeq H$ and $H \simeq J$, then $G \simeq J$ (composition)

[1]

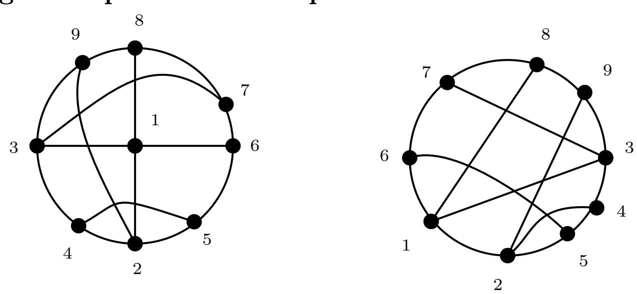


Consider another example.

[1] That's why Prof. Wagner likes to teach advanced section. This takes 5 minutes in regular section, however, it takes only 1 minute here...



Degree Sequence of a Graph 4 4 4 3 3 3 3 3 3



adjacency table

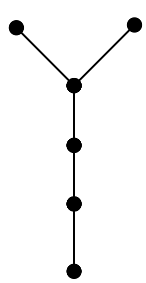
v	neighbours of v
1	2 3 6 8
2	1 4 5 9
3	1 4 7 9
4	2 3 5
5	2 4 6
6	1 5 7
7	3 6 8
8	1 7 9
9	2 3 8

Let's consider the degree of a vertex v.

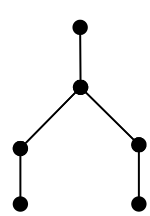
Definition $\deg(v)$ = number of edges incident to it

If $f : G \to H$ is an isomorphism, then $\forall v \in V(G), \deg_H(f(v)) = \deg_G(v)$.

However the converse is not true.

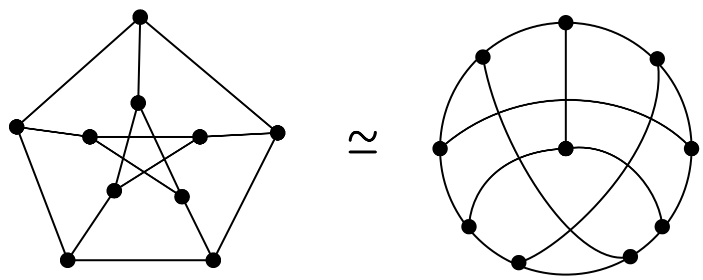


$\not\cong$



Subgraphs[2] induced by vertices degree 4.

Now consider this.



Peterson Graph

$p = |V| = 6, \qquad q = |E| = 10$
degree sequence: $4+4+4+3+3+2 = 20 = 2|E|$, this leads to handshake lemma (degree sum formula).

6.1 Handshake Lemma

For any graph $G = (V, E)$, $\sum\limits_{v \in V} \deg(V) = 2|E|$

Proof Let $S = \{(v, e) : v \in V, e \in E, v \in e\}$. Each edge contains 2 vertices $|S| = 2|E|$. Each vertex $v \in V$ is contained in $\deg(v)$ edges. So $|S| = \sum_{v \in V} \deg(V) = 2|E|$. □

Eg Find a graph with degree sequence

$$5\ 4\ 4\ 4\ 3\ 3\ 3\ 3\ 2\ 2$$

$$\sum_{v \in V} \deg v = 33 \implies |E| = 16\frac{1}{2} \qquad \implies \text{DNE}$$

This leads to a corollary.

Corollary Every graph has an even number of vertices of odd degrees.

[2] formal definitions later

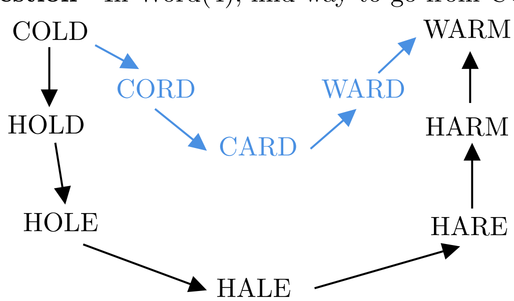
7 Connectedness

$G = (V, E)$ is "connected" if for any two vertices $v, w \in V$, you can "walk" along the edges of G to get from v to w.

Eg Word graphs Word(n).

> **Vertices** n-letter words in English (except: no swear words)
>
> **Edges** join words that differ in exactly one letter

Question In Word(4), find way to go from COLD TO WARM



Is ~~this~~ this as short as possible? Yes

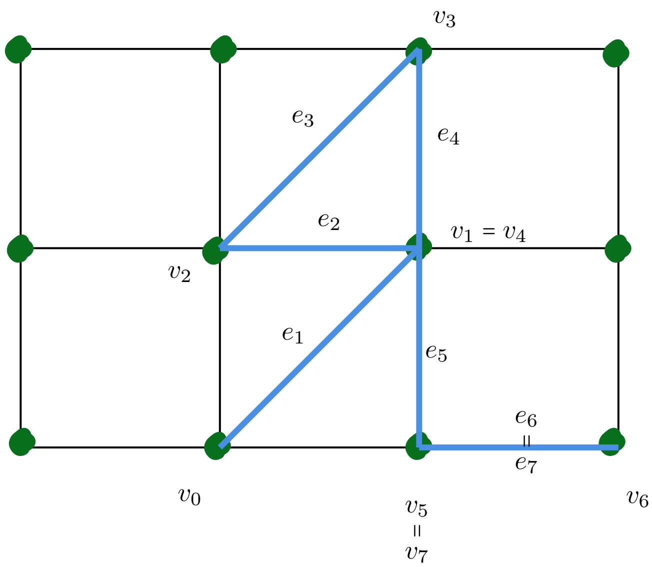
Exercise Try these if you like:

- FISH $\rightsquigarrow$ BIRD
- PINK $\rightsquigarrow$ BLUE

Definition A walk in $G = (V, E)$ is a sequence of vertices & edges.

$$w: \quad v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$$

each $v_i \in V$ for $0 \le i \le k$
each $e_i = \{v_{i-1}, v_i\}$ for $1 \le i \le k$.



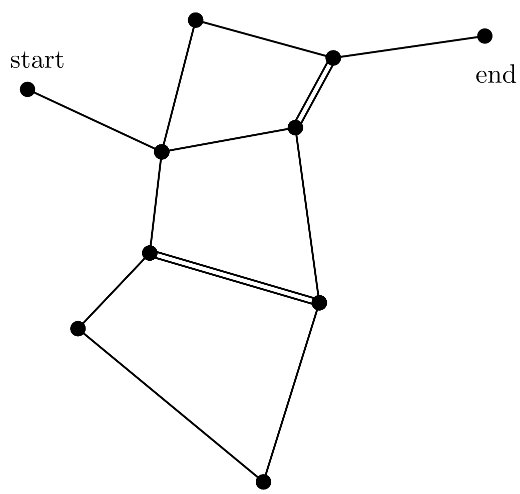
7.1 Walks, Trails, Paths

Recall the definition for walk, and we have two more definitions:

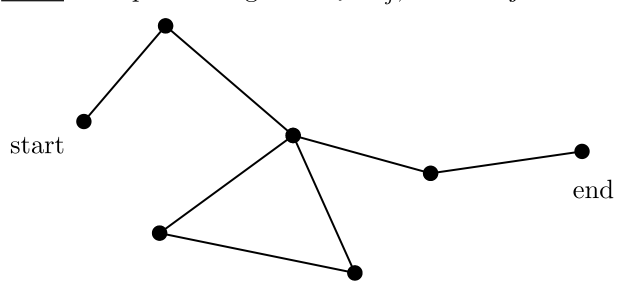
- Walk:

$$w: \quad v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$$

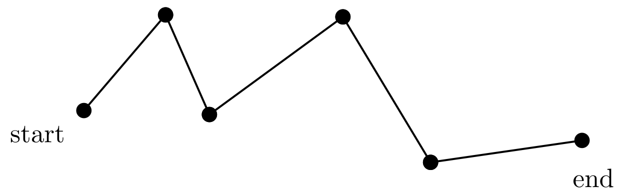
  each $v_i \in V$ for $0 \leq i \leq k$
  each $e_i = \{v_{i-1}, v_i\}$ for $1 \leq i \leq k$.



- Trail: no repeated edges. if $e_i = e_j$, then $i = j$.



- Path: no repeated vertices. If $v_i = v_j$, then $i = j$.



Proposition If there is a walk from v to w in $G = (V, E)$, then there is a path from v to w in G.

Proof Let $W : v_0 e_1 v_1 \ldots v_{k-1} e_k v_k = w$ be a walk form v to w with as few edges as possible.
Claim: W is a path.
Suppose $v_i = v_j$ with $i < j$.
Then

$$\begin{array}{ll} v_0 e_1 \ldots v_{i-1} e_i v_i & \\ \qquad\qquad\qquad \| & \\ \qquad\qquad\qquad v_j e_{j+1} v_{j+1} \ldots e_k v_k & \end{array}$$

is a walk form v to w that is shorter than W. Contradiction □

7.2 Reaches

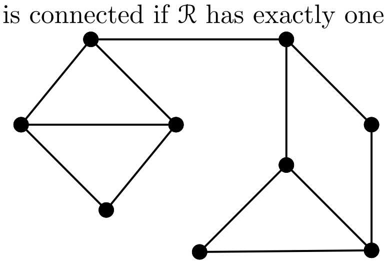
Let $G = (V, E)$ be a graph.
Define a binary relation $\mathcal{R}$ on V called "reaches".
For $v, w \in V$: v reaches w if and only if there is a walk form $v = v_0$ to $v_k = w$.

$\mathcal{R}$ is a equivalence relation

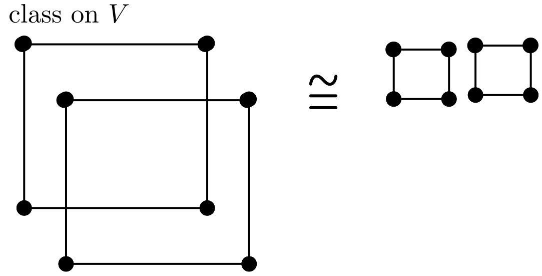
- $v \ \mathcal{R} \ v$ (soft potato)
- if $v \ \mathcal{R} \ w$, then $w \ \mathcal{R} \ v$
- if $v \ \mathcal{R} \ w$, and $w \ \mathcal{R} \ z$, then $v \ \mathcal{R} \ z$

7.3 Connected

G is connected if $\mathcal{R}$ has exactly one equivalence class on V



Connected

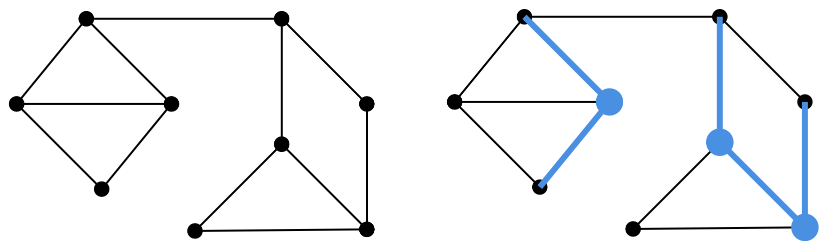


Not Connected

7.4 Subgraphs

Let $G = (V, E)$ be a graph. A subgraph of G is $H = (W, F)$ with

- $W \subseteq V$
- $F \subseteq E$
- (W, F) is a graph (Every edge in F needs both ends in W).



$H = (W, F)$

not a subgraph

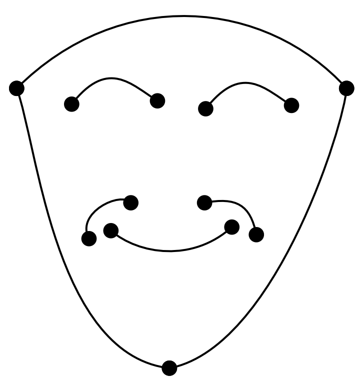
It is not a graph
due to edges to
a nonexisting vertex.

7.5 Induced Subgraph

$H = (W, F)$

- Pick $W \subseteq V$
- Then $F = \{e \in E, e \subseteq W\}$ (all edges of G with both ends in W)

A (connected) component of $G = (V, E)$ is a subgraph induced by an equivalence class of $\mathcal{R}$.

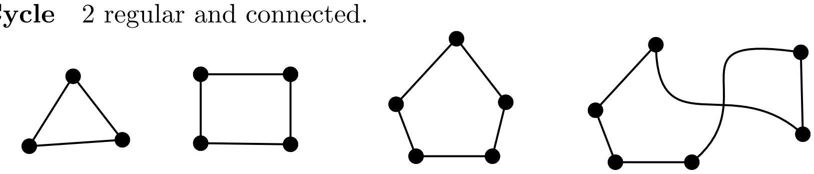


6 components

7.6 Regular

k-regular Every vertex has degree k.

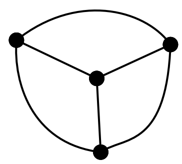
Cycle 2 regular and connected.



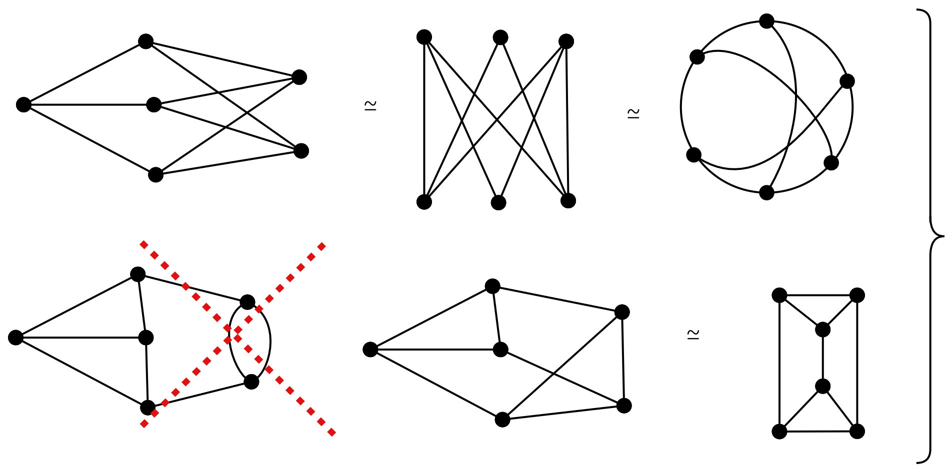
C_3 C_4 C_5 C_6

7.7 Connected 3-regular Graph

$p = 4$



$p = 6$

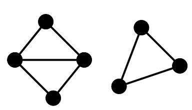


the only two

Definition Let $C(G) = \#$ of connected components of G.

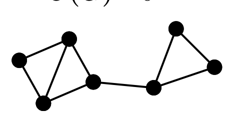
- $C(G) = 1$ means <u>connected</u>
- $C(G) = 0$? (not connected since no components [1]) empty graph $K_0 = (\varnothing, \varnothing)$

$C(G) = 1$



not connected

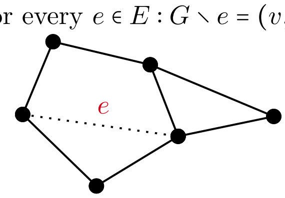
$C(G) = 0$



connected

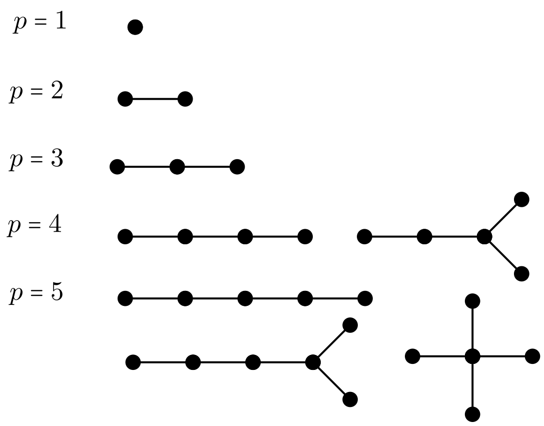
7.8 Minimally Connected Graphs

$G = (V, E)$ is connected
For every $e \in E : G \setminus e = (v, E \setminus \{e\})$ is not connected.



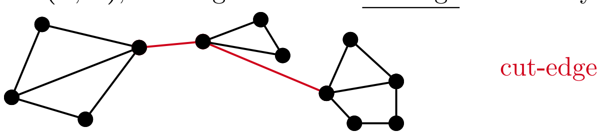
Example $p = |V|$.

[1] arguable



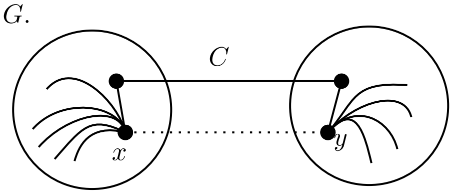
7.9 Cut-edge

In $G = (V, E)$, and edge $e \in E$ is a cut-edge if and only if $C(G \setminus e) > C(G)$



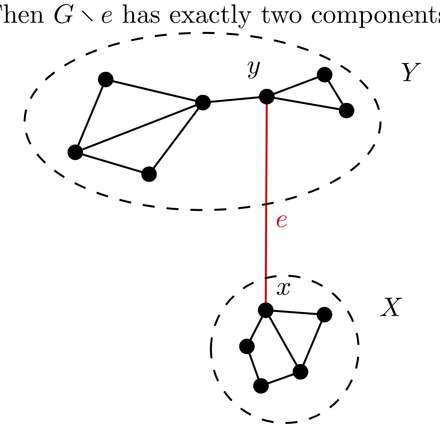
G is minimally connected if and only if it is connected and every edge is a cut-edge.

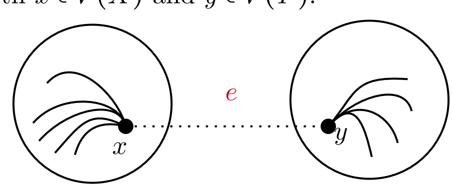
Proposition For $G = (V, E)$, and edge $e \in E$ is a cut-edge if and only if e is not in a cycle (subgraph) of G.



Proof See math 239 notes... $\square$

Theorem Let $G = (V, E)$ be a connected graph.
Let $e \in E$ be a cut-edge: $e = \{x, y\}$.
Then $G \setminus e$ has exactly two components. X and Y with $x \in V(X)$ and $y \in V(Y)$.





cut-edge (this course)
bridge (math 239)
isthmus (graph theory paper before 1970s)

Proof
Let X be the component of $G \setminus e$ induced by vertices reachable from x in $G \setminus e$.
Let Y be the component of $G \setminus e$ induced by vertices reachable from y in $G \setminus e$.

We need to show:

- $X \neq Y$
- every vertex $z \in V$ is either in X or in Y

Now prove by contradiction:

- Suppose that $X = Y$.
  There is a walk from X to Y in $G \setminus e$.
  There is a path P from X to Y in $G \setminus e$.
  Now $C = P \cup \{e\}$ is a cycle in G containing e. So e is not a cut-edge of G. Contradiction.
- Let $z \in V(G)$. Since G is connected, there is a walk from x to z, so there is a path from x to z:

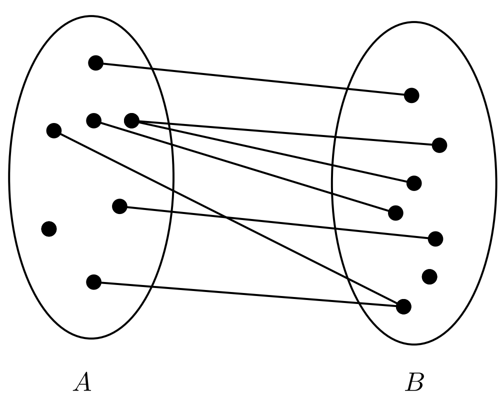
$$P : \quad x = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k = z$$

  - If P does not use e, then P is a path in X, so $z \in V(X)$.
  - If P does use e, then $e_1 = e$ and $v_1 = y$ and $z \in V(Y)$.

□

7.10 Bipartite Graph

A graph $G = (V, E)$ is bipartite, if one can write $V = A \cup B$ with $A \cap B = \varnothing$ and every edge $e \in E$ has one end in A and one end in B.



7.10.1 Handshake Lemma for bipartite graphs

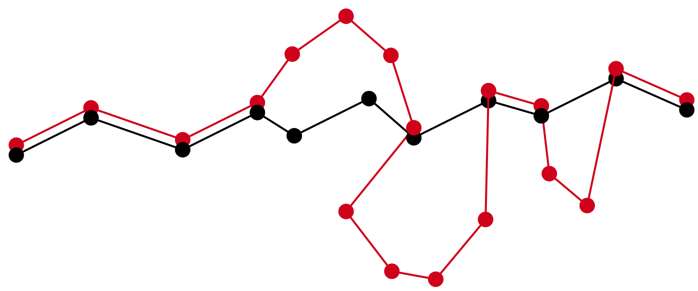
Let $G = (V, E)$ be a graph with bipartition (A, B). Then

$$\sum_{v \in A} \deg(v) = |E| = \sum_{v \in B} \deg(v)$$

7.11 Things to remember...

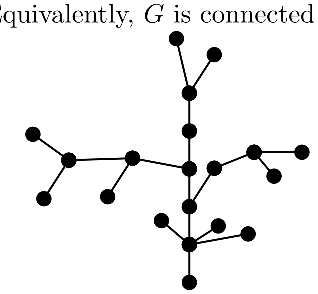
- Handshake Lemma
- Definitions

- If there is a walk form v to w, then there is a path form v to w.
- If $e \in E(G)$ is a cut-edge of connected graph, then $e = \{x, y\}$ and $G \smallsetminus e$ has exactly 2 components X and y with $x \in V(X)$ and $y \in V(Y)$.
- $e \in E(G)$ is a cut-edge if and only if it is not contained in a cycle of G.
- If there are two different paths form v to w in G, the G contains a cycle.



8 Trees

G is minimally connected if G is connected and every edge is a cut-edge.
Equivalently, G is connected and has no cycles. $\longrightarrow$ Definition of a tree.

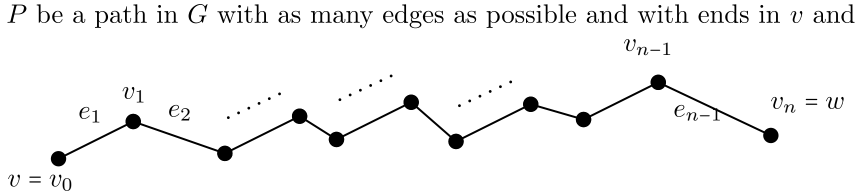


Small Trees

$p = \lvert V \rvert$	trees
1	see
2	the
3	previous
4	section
5	7.8

Proposition Let $G = (V, E)$ with $P = |V| \geq 2$ vertices and G is a tree. Then G has at least two vertices of degree 1.

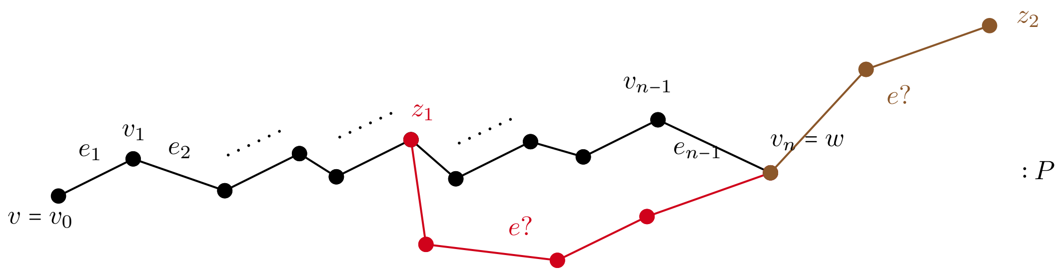
Proof (External Method)
Let P be a path in G with as many edges as possible and with ends in v and w.



$: P$

Since $p \geq 2$, we have $v \neq w$

Claim Both v, w have degree 1.
Suppose that $e = \{w, z\}$ is an edge not in P.

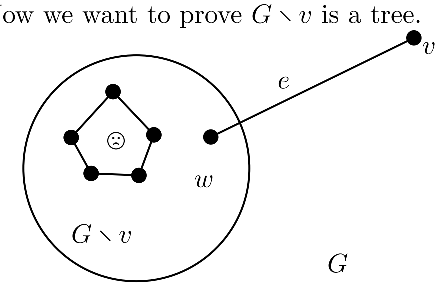


- If z is on P, then $P \cup \{e\} \subseteq G$ contains a cycle. Contradiction.
- If z is not on P, then P is not as long as possible. Contradiction.

□

Proposition If $G = (V, E)$ is a tree with $p \geq 1$ vertices, then G has $q = p - 1$ edges.

Proof Basis of induction $p \leq 5$.
Assume the result for trees with $p - 1 \geq 1$ vertices.
Let G be a tree with $p \geq 2$ vertices.
Then G has a vertex v of degree 1, incident with edge $e = \{v, w\}$.
Now we want to prove $G \setminus v$ is a tree.



- $G \setminus v$ has $p - 1$ vertices.
- $G \setminus v$ is connected because $w \mathcal{R} z$ for all $z \in V(G)$ in G. So $w \mathcal{R} z$ for all $z \in V(G \setminus v)$ in $G \setminus v$. So $G \setminus v$ is connected.
- $G \setminus v$ contains no cycles. Since $G \setminus v$ is a subgraph of G and G has no cycles.

By induction we have

$$q(G \setminus v) = p(G \setminus v) - 1 = p - 2$$

So $q(G) = 1 + q(G \setminus v) = p - 1 = p(G) - 1$ □

8.1 The 2-out-of-3 Theorem

Let $G = (V, E)$ be a graph with $p = |V|$ and $q = |E|$. Consider the following three conditions:

(i) G is connected

(ii) G has no cycles

(iii) $q = p - 1$

Any two of these conditions implies the other one.

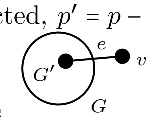
Proof

(i)&(ii) $\Longrightarrow$ (iii) Done on Wednesday

(i)&(iii) $\Longrightarrow$ (ii) Assume that G is connected and $q = p - 1$.
We can assume $p \geq 3$ (otherwise G is $\bullet$ or $\bullet\!\!-\!\!\bullet$)

> <u>Claim:</u> G has a vertex of degree 1.
> Suppose not. Every vertex has degree ≥ 2. Since G is connected and $p \geq 3$. By handshake lemma
>
> $$2q = 2|E| = \sum_{v \in V} \deg(v) \geq 2|V| = 2p$$
>
> So $q \geq p$. contradiction.

Let $v \in V$ has degree 1, incident with $e \in E$, and consider $G' = G \setminus v$. G' is connected, $p' = p - 1$, and $q' = q - 1$, so $q' = p' - 1$. By induction (on p or q), G' has no cycles, so G has no cycles.

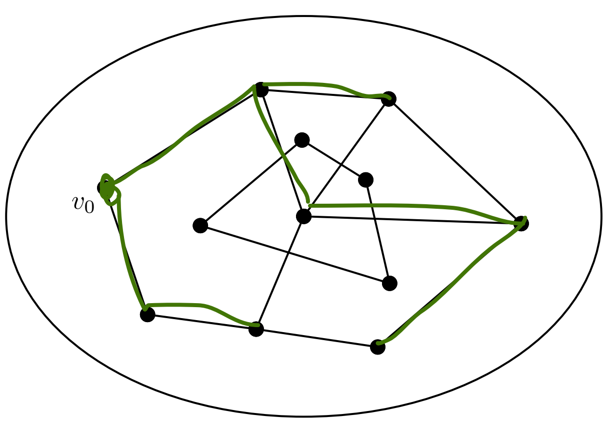


(ii)&(iii) $\Longrightarrow$ (i) Assume G has no cycles and $q = p - 1$. Assume $p \geq 3$, too.
Let the connected components of G be $G_1, G_2, \ldots, G_c$. We will show that $c = 1$.
Let $p_i = |V(G_i)|$ and $q_i = |E(G_i)|$.
Each G_i is a tree. So by (i) & (ii) $\Longrightarrow$ (iii), each $q_i = p_i - 1$.
Now $p = p_1 + \ldots + p_c$, $q = q_1 + \ldots + q_c$.
So

$$1 = p - q = (p_1 - q_1) + \ldots + (p_c - q_c) = c$$

$\square$

Next Class



8.2 Spanning Trees

Let $G = (V, E)$ be a graph. A spanning tree for G is a subgraph $H = (W, F)$ such that

- H is spanning ($W = V$), and
- H is a tree.

Proposition G has a spanning tree if and only if G is connected.

Proof If T is a spanning tree in G, then $v \mathcal{R} w$ in T for all $v, w \in V$, so $v \mathcal{R} w$ in G for all $v, w \in V$, so G is connected.
Conversely, assume that G is connected. G is connected, so it has connected spanning subgraphs.
Let H be a connected subgraphs of G with as few edges as possible.

- H is spanning and connected. ✓
- Claim: Every edge of H is a cut-edge.

Proof of the claim Suppose not. If $e \in E(H)$ is not a cut-edge, then $H \setminus e$ is a connected spanning subgraph with $q(H \setminus e) < q(H)$. Contradition.

- So H has no cycle.
- So H is a spanning tree of G.

□

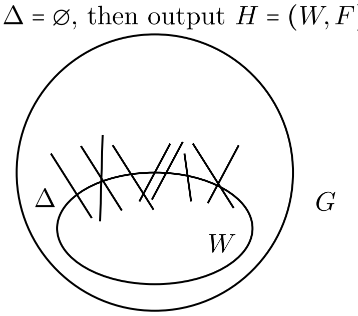
Corollary If G is connected, then $q \geq p - 1$.

8.3 Search Trees

Input graph $G = (V, E)$ and "root" vertex $v_0 \in V$.

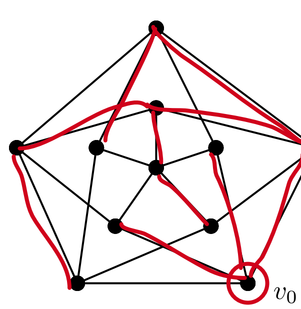
Initialization $W = \{v_0\}$ and $F = \varnothing$.

$(*)$ Let $\Delta = \{e \in E : e \text{ has one end in } W \text{ and one end in } V \setminus W\}$
If $\Delta = \varnothing$, then output $H = (W, F)$ and STOP.



Otherwise, choose any $e = \{x, y\} \in \Delta$ with $x \in W$ and $y \in V \setminus W$. Update. $W \leftarrow W \cup \{y\}$ and $F \leftarrow F \cup \{e\}$. GoTo $(*)$

Example



Claim Upon output, $T = (W, F)$ is a spanning tree for the component of G containing v_0

Proof

(1) T is connected and $q(T) = p(T) - 1$. By induction on the number of iteration of the loop.
Basis: $W = \{v_0\}, F = \varnothing$: $\bullet_{v_0}$ clear
Step: Assume (W, F) is a tree. $e = \{x, y\} \in \Delta$ (with $x \in W$) and $W' = W \cup \{y\}, F' = F \cup \{e\}$.

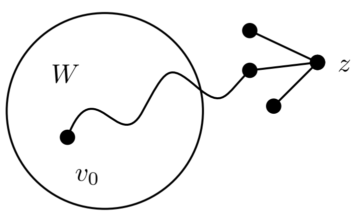
Claim (W', F') is a tree.

Proof

- Then $q' = q + 1 = p = p' + 1$
- (W', F') is connected

Now $x \mathcal{R} w$ in (W, F) for all $w \in W$, and $x \mathcal{R} y$ in (W', F'). So (W', F') is connected.
By 2-out-of-3 Theorem, (W', F') is a tree.

(2) Every vertex in W is reachable from v_0 in (W, F), so in G, so is the component of G containing v_0. Conversely, suppose that $z \in V$ is in the component of G containing v_0, but not in W upon output. There is a path from v_0 to w in G

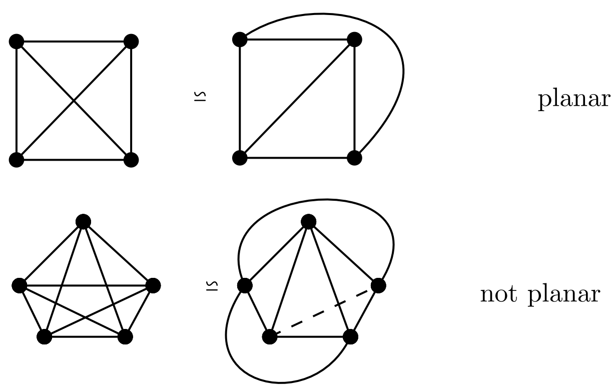


$$P: \quad \underset{\in W}{v_0} \quad e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k \underset{\notin W}{v_k = z}$$

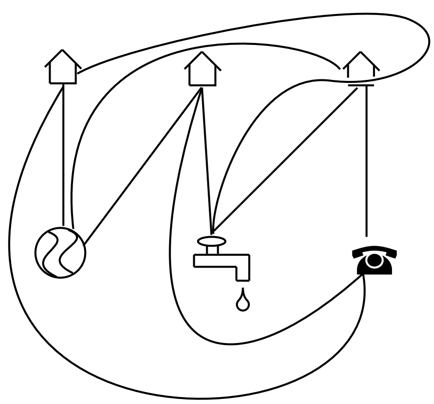
Now there must be an edge $e_i\{v_{i-1} v_i\}$ of P with $v_{i-1} \in W$, and $v_i \notin W$. So $e_i \in \Delta$, so $\Delta \neq \varnothing$, so the algorithm hasn't stopped. Contradiction. $\square$

9 | Planar Graphs

Can de drawn without “crossing edges”



Definition Can be drawn in the plane without crossing edge.



9.1 Plane embedding of a graph

$G = (V, E)$ a graph.

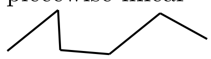
$P = \{P_v : v \in V\}$ a set of distinct points in $\mathbb{R}^2$ (represents vertices)

$\Gamma = \{\gamma_e : e \in E\}$ a set of continuous curves in $\mathbb{R}^2$ (represents edges)

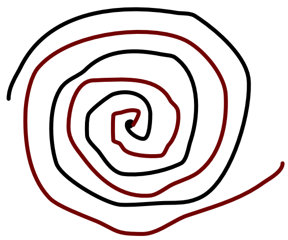
- $\gamma_e : [0, 1] \to \mathbb{R}^2$ is
  - injective (non-self-crossing), and

- not "weird"
  - differentiable to all orders (continuously) C^∞

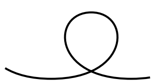
  - piecewise linear



point is to avoid

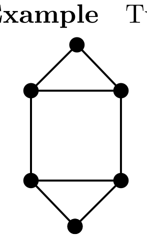


- For each $e = \{v, w\} \in E, \{\gamma_e(0), \gamma_e(1)\} = \{P_v, P_w\}$
  γ_e has ends P_v and P_w
- For $e \neq f$ in E. If $z \in \mathrm{Im}(\gamma_e) \cap \mathrm{Im}(\gamma_f)$, then $z = P_v$ where $\{v\} = e \cap f$ (edges meet only at end points)

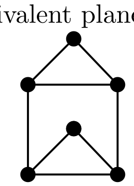


bad!

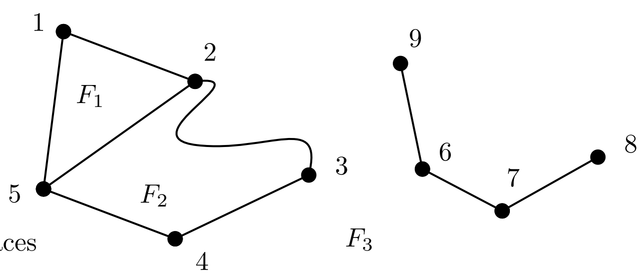
Example Two inequivalent plane embeddings of the same graph.



$\simeq$



9.2 Face of a planar embedding

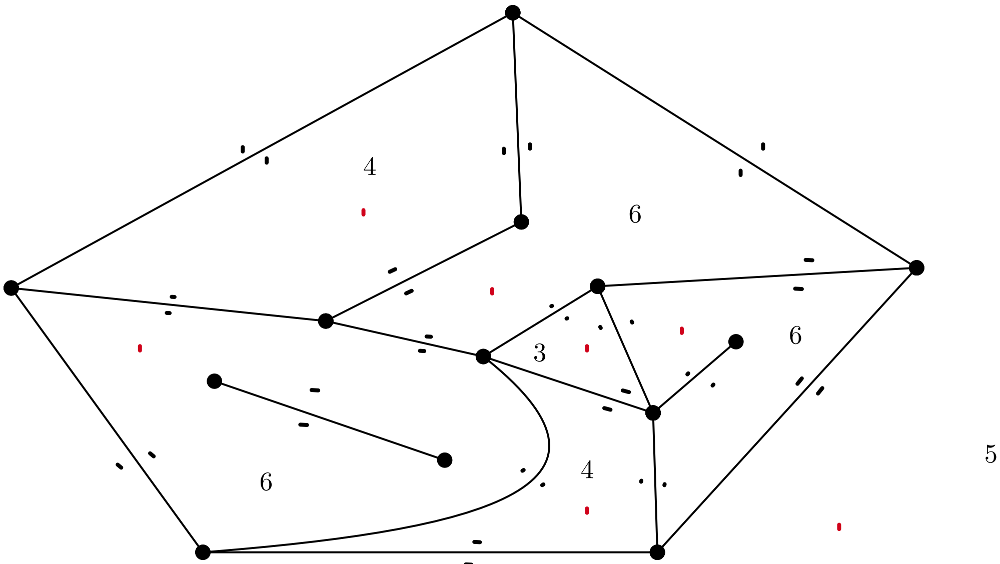


Three faces

$G = (V, E)$ a graph
(P, Γ) a plane embedding of G.
Cut $\mathbb{R}^2$ along all curves in Γ.
It falls into pieces - these are the <u>faces</u> of the embedding (P, Γ)

9.2.1 Degree of a face

Example



seven faces

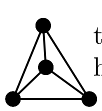
- each edge is a wall
- put 2 cookies on top of each wall
- minor earthquake - one cookie falls one each side
- degree of a face = number of cookies you can eat

9.2.2 Handshake Lemma for Faces

$$\sum_{\text{faces } F} \deg(F) = 2|E|$$

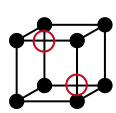
For a plane embeddings r = number of faces

Eg



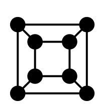
tetra-
hedron

$p = 4$
$q = 6$
$r = 4$



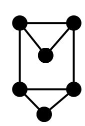
bad!
no r

$\simeq$

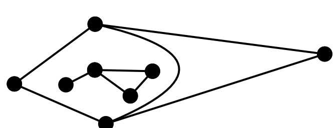


cube

$p = 8$
$q = 12$
$r = 6$



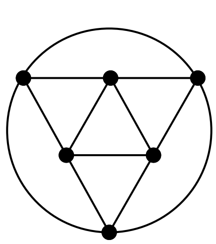
$p = 6$
$q = 8$
$r = 4$



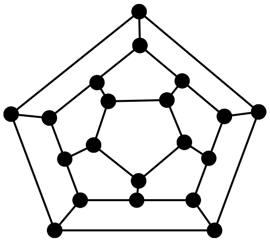
$p = 8$
$q = 9$
$r = 4$

These lead to Euler's formula. Let c be the number of components

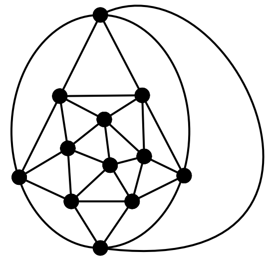
$$p - q + r = c + 1$$



Octahedron

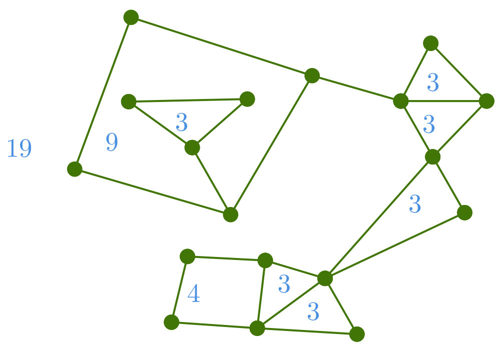


Dodecahedron



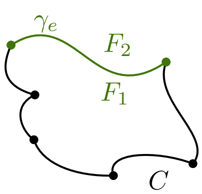
Icosahedron

9.3 Cut-edges in plane graphs

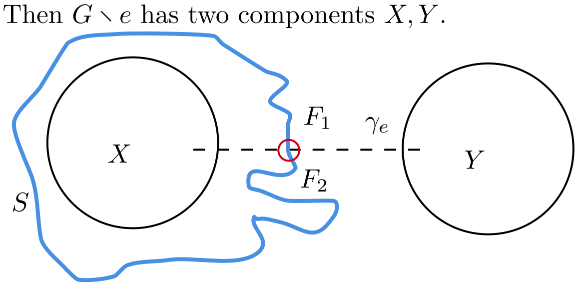


Lemma Let $G = (V, E)$ be a plane graph (Embedding (P, Γ))
Let $e \in E$. Let F_1 and F_2 be the two faces on either side of γ_e. Then $F_1 = F_2$ if and only if e is a cut-edge.

Proof If e is a cut-edge then it is in a cycle C. The curves $\{\gamma_c : c \in E(C)\}$ form a simple closed curve with one of F_1, F_2 inside and the other outside. So $F_1 \neq F_2$.



If e is a cut-edge. Reduce to the case that G is connected. (Exercise)
Then $G \setminus e$ has two components X, Y.



There is a simple closed curve S with the drawing of X inside S and the drawing of Y outside, intersecting γ_e in one point.

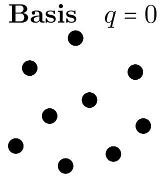
Now the image of S not in the image of γ_e is contained in a single face of the embedding (P, Γ) of G. So $F_1 = F_2$. ☐

9.4 Euler's Formula

Let $G = (V, E)$ be a graph with plane embedding (P, Γ). p vertices, q edges, r faces, c components.
Then $p - q + r = c + 1$

Proof By induction on q.

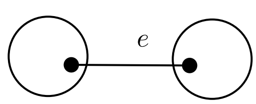
Basis $q = 0$



$$\begin{cases} p = \\ q = 0 \\ r = 1 \\ c = p \end{cases} \qquad \text{so } p - q + r = p - 0 + 1 = c + 1 \qquad \checkmark$$

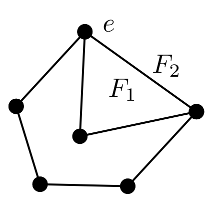
Induction Step Assume the result for graphs with $q - 1$ edges. Let G have $q \geq 1$ edges. Let $e \in E$. Two cases: e is/isn't a cut-edge.
Let $G' = G \setminus e$ and define p', q', r', c' accordingly.
So $p' - q' + r' = c' + 1$ by induction, since $q' = q - 1$.

- <u>case:</u> e is a cut-edge
  $p' = p$
  $q' = q - 1$
  $r' = r$
  $c' = c + 1$



$$\begin{aligned} p - q + r &= p' - (q' - 1) + r' \\ &= p' - q' + r' - 1 \\ &= c' + 1 - 1 \\ &= c' = c + 1 \qquad ☺ \end{aligned}$$

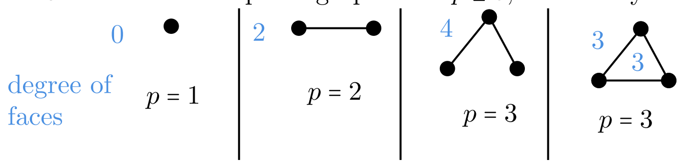
- <u>case:</u> e is not a cut-edge.
  $p' = p$
  $q' = q - 1$
  $r' = r - 1$
  $c' = c$



$$\begin{aligned} p - q + r &= p' - (q' - 1) + (r' - 1) \\ &= p' - q' + r' \\ &= c' + 1 = c + 1 \qquad ☺ \end{aligned}$$

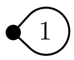
$\square$

Fact G is a connected plane graph with $p \geq 3$, then every face has degree ≥ 3.



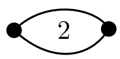
larger p

- faces of degree 0
  $\implies$ no edges
  $\implies$ not connected
- face of degree 1:



not allowed

- face of degree 2:



not allowed

Proposition Let $G = (V, E)$ be a connected graph with $p \geq 3$ vertex and q edges. If G is planar then $q \leq 3p - 6$.

Proof Let (P, Γ) be a plane embedding of G with r faces ($c = 1$).
Handshake Lemma for Faces

$$2q = \sum_{\text{faces } F} \deg(F) \geq \sum_{\text{faces } F} 3 = 3r \qquad \text{by the Fact above}$$

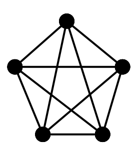
By Euler's Formula

$$p - q + r = c + 1 = 2$$
$$3p - 3q + 3r = 6 \leq 3p - 3q + 2q = 3p - q$$

So $q \leq 3p - 6$ ☐

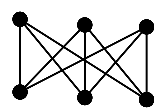
Eg

K_5



connected ✓
$p = 5 \geqslant 3$ ✓
$q = 10$

$K_{3,3}$



connected ✓
$p = 6 \geqslant 3$ ✓
$q = 9$

$q = 10 > 9 = 3p - 6$
So K_5 is not planar

$q = 9 \nleqslant 12 = 3p - 6$
no conclusion

$q \leq 3p - 6$ with equality if and only if every plane embedding, every face has degree 3.

Proposition Let G be planar, connected, $P \geq 3$, no 3-cycles, q edges. Then $q \leq 2p - 4$.

Proof Embed G in the plane. Every face has degree ≥ 4. By handshake lemma for faces

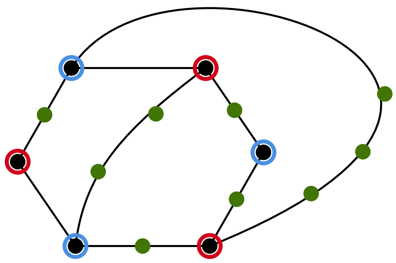
$$2q = \sum_{\text{faces } F} \geq 4r$$

So $q \geq 2r$. Then by Euler's Formula, $p - q + r = c + 1 = 2$

$$4 = 2p - 2q + 2r \leq 2p - 2q + q = 2p - q$$

Hence $q \leq 2p - 4$ ☐

9.5 Subdivision

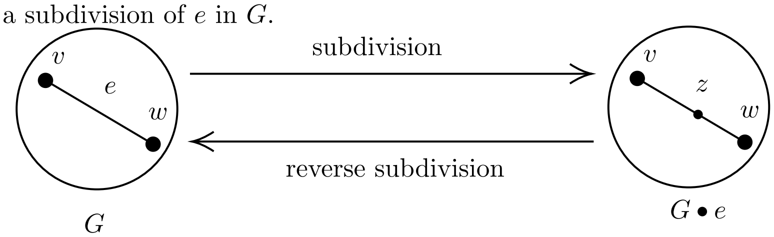


repeated subdivision of $K_{3,3}$

Definition Subdivision of an edge $e \in E$ in a graph $G = (V, E)$.
Let $e = \{v, w\} \in E$. Let z be a "new" vertex not in V.

$$G \bullet e = \{V \cup \{z\}, (E \setminus \{\{v,w\}\} \cup \{\{v,z\},\{w,z\}\})\}$$

is a subdivision of e in G.



Lemma G is planar if and only if $G \bullet e$ is planar.

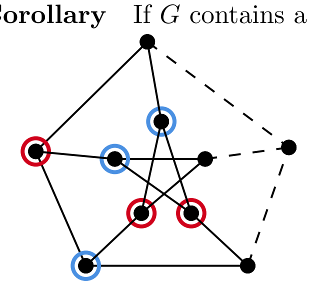
Proof Exercise. [1] □

Corollary If G,H are related by a sequence of subdivisions and reverse subdivisions, the G is planar if and only if H is planar.

Lemma If G contains a subgraph H and H is not planar, then G is not planar.

Proof Exercise. (use contradiction) □

Corollary If G contains a (repeated) subdivision of K_5 or $K_{3,3}$, then G is not planar.



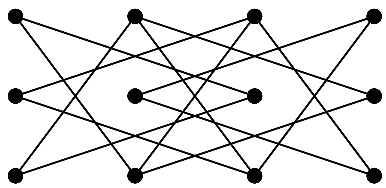
subdivision of $K_{3,3}$ in Peterson
Peterson is not planar

9.6 Kuratowski's Theorem

G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

[1] Prof's favourite proof technique

Proof in CO 342 □



$KM(3,4)$
(planar)

9.7 "Numerology" of planar graphs

Let $G = (V, E)$ be planar, connected, $p \geq 3$ vertices, q edges. So $q \leq 3p - 6$ and we know when equality holds.
Let n_k be the number of vertices of degree k, for each $k \in \mathbb{N}$. Then $n_0 = 0$. Now

$$p = n_1 + n_2 + n_3 + \ldots = \sum_{k=1}^{\infty} n_k$$

$$6p - 12 \geq 2q = \sum_{v \in V} \deg(v) = n_1 + 2n_2 + 3n_3 + \ldots = \sum_{k=1}^{\infty} k \cdot n_k$$

$$n_1 + 2n_2 + 3n_3 + \ldots + 6n_6 + 7n_7 + \ldots \leq 6n_1 + \ldots + 6n_6 + 7n_7 + \ldots - 12$$

Hence we have a formula:

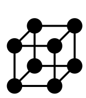
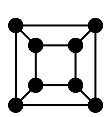
$$n_7 + 2n_8 + 3n_9 + \ldots + 12 \leq 5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5$$

There must be a vertex of degree ≤ 5.

- Icosahedron: $p = n_5 = 12$
- Octahedron: $p = n_4 = 6$

9.8 Platonic Solids

- connected graph drawn in the plane
- every vertex has degree d ($d \geq 0$, but case $0, 1, 2$ are kind of degenerate)
- every face has degree k ($k \geq 0$, but small k is "degenerate")

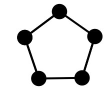
cube
$d = 3$
$k = 4$

From Monday
every connected planar graph has a vertex of degree ≤ 5

9.8.1 Case Analysis

$d = 0$  $k = 0$ $p = 1$, $q = 0$, $r = 1$

$d = 1$  $k = 2$ $p = 2$, $q = 1$, $r = 1$

$d = 2$  any $k \geq 3$ $p = k$, $q = k$, $r = 2$

↑ Above are degenerate cases —— Below are general cases ↓

Case $d = \{3, 4, 5\}$

$$\left.\begin{array}{ll}\text{Handshake Lemma:} & 2q = pd \\ \text{Face Kissing Lemma:} & 2q = rk \\ \text{Euler's Formula:} & p - q + r = 2\end{array}\right\} \implies \frac{2q}{d} - q + \frac{2q}{k} = 2 \implies \boxed{\frac{2}{d} + \frac{2}{k} = 1 + \frac{2}{q} > 1}$$

$d = 3$

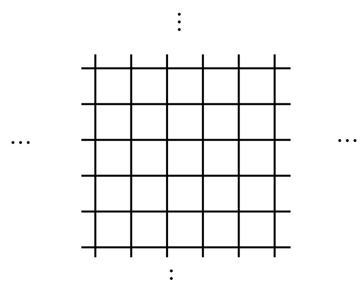
- $k = 3$ $\quad \frac{2}{3} + \frac{2}{3} = \frac{4}{3} = 1 + \frac{2}{6} > 1$ $\quad q = 6$ — tetrahedron
- $k = 4$ $\quad \frac{2}{3} + \frac{2}{4} = 1 + \frac{2}{12} > 1$ $\quad q = 12$ — cube
- $k = 5$ $\quad \frac{2}{3} + \frac{2}{5} = 1 + \frac{2}{30} > 1$ $\quad q = 30$ — dodecahedron
- $k \geq 6$ $\quad \frac{2}{3} + \frac{2}{k} \leq \frac{2}{3} + \frac{2}{6} = 1$ — no examples

$d = 4$

- $k = 3$ $\quad \frac{2}{4} + \frac{2}{3} = 1 + \frac{2}{12} > 1$ $\quad q = 12$ — octahedron
- $k \geq 4$ $\quad \frac{2}{4} + \frac{2}{k} \leq \frac{2}{4} + \frac{2}{4} = 1$ — no examples

$d = 5$

- $k = 3$ $\quad \frac{2}{5} + \frac{2}{3} = 1 + \frac{2}{30} > 1$ $\quad q = 30$ — icosahedron
- $k \geq 4$ $\quad \frac{2}{4} + \frac{2}{k} \leq \frac{2}{5} + \frac{2}{4} = \frac{18}{20} < 1$ — no examples
  We can have infinite vertices this case...



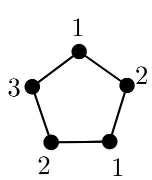
9.9 Colouring Graphs

Example Let's define a Graph

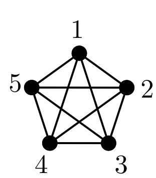
- vertices: radio stations
- edges: $\{v, w\} \in E$ means station v interferes with station w if they are in the same frequency.

Problem

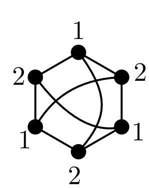
- Assign frequencies to station so that no two stations interfere
- use as few frequencies as possible



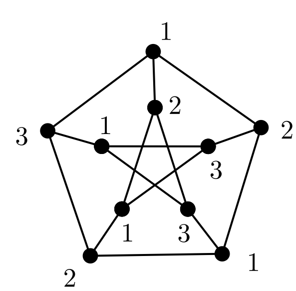
needs 3



needs 5



needs 2
bipartite (with $\geqslant 1$ edge)



needs 3

Definition A k-colouring of a graph is a graph $G = (V, E)$ in a function $f : V \to \{1, 2, \dots, k\}$ for some $k \in \mathbb{N}$

Proper A colouring is proper if for every $e = \{e, w\} \in E : f(v) \neq f(w)$.
The smallest k for which G has a proper k-colouring is chromatic number of G, denoted $\chi^{(G)}$.

$$\chi^{(G)} \leq |V| \quad \text{so it exists}$$

Which graphs have chromatic number k?

- $k = 0$: no vertex
- $k = 1$: graphs with no edges, at least one vertex
- $k = 2$: bipartite with at least one edge
- $k = 3$: Given a non-bipartite graph G, to determine whether $\chi^{(G)} = 3$ is NP-complete

  Let the components of G be $G_1, G_2, \dots, G_c$

$$\text{Then} \quad \chi^{(G)} = \max_{1 \leq i \leq c} \chi^{(G_i)} \tag{$*$}$$

$\boxed{\text{Assume every vertex has degree} \leq \Delta\text{, then } \chi^{(G)} \leq \Delta + 1}$

Proof Induction on $p = |V|$. If $p \leq \Delta + 1$, give every vertex its own colour.

Induction Step Let $v \in V$. Consider $G \setminus v$ has $p - 1$ vertices and all of degree $\leq \Delta$. By induction, $G \setminus v$ has a proper colouring $f : V \setminus \{v\} \to \{1, 2, \dots, \Delta + 1\}$. Now v has at most Δ neighbours in $V \setminus \{v\}$ and they use at most Δ colours. Since we have $\Delta + 1$ colours available, there is a new colour we can assign a to v to get a proper colouring. □

Note $\chi^{(K_p)} = p$

9.9.1 Six Colour Theorem

If G is planar, then $\chi^{(G)} \leq 6$

Proof Induction on $p = |V|$

- Basis: If $p \leq 6$, then give each vertex its own colour.
- Induction step: (By $(*)$, we can assume that G is connected.[2])
  Since G is planar, G has a vertex v of degree ≤ 5. $G \setminus v$ has a proper 6-colouring $f : V \setminus \{v\} \to \{1, 2, \dots, 6\}$. By induction, now v has only 5 neighbours, so there is some colour in $\{1, 2, \dots, 6\}$ that can be used at v to extend f to a proper colouring of G.

□

9.9.2 Four Colour (Map) Theorem

Francis Guthrie ~ 1850

- Map of the British Empire
- Regions could be coloured so that bordering regions has different colours using at most four colours.
- Is this true for any (plane) map?

  He is colouring regions (faces).

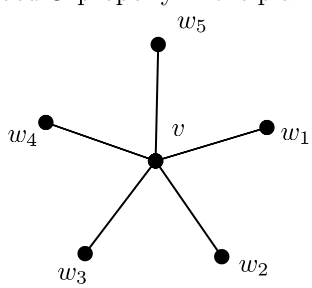
[2] It seems we don't have to use this.

9.9.3 Five Colour Theorem

If G is a planar graph, then G can be properly coloured with at most 5 colours.

Proof Induct on $p = |V(G)|$. Basis $p \leq 5$, then give every vertex its own colour.
Let G be a planar graph with $p \geq 6$ vertices. We can assume that G is connected.
G has a vertex of degree 1,2,3,4 or 5.
• If G has a vertex v of degree ≤ 4, then consider $G \setminus v$.
Then $G \setminus v$ is planar, so we can induct.
Let $f : V \setminus \{v\} \to \{1,2,3,4,5\}$ be a proper 5-colouring of $G \setminus v$. Since $\deg_G(v) \leq 4$, there is at least one "colour" $c \in \{1,2,3,4,5\}$ not used by f on the neighbours of v. Set $f(v) = c$ to extend f to a proper five-colouring of G.

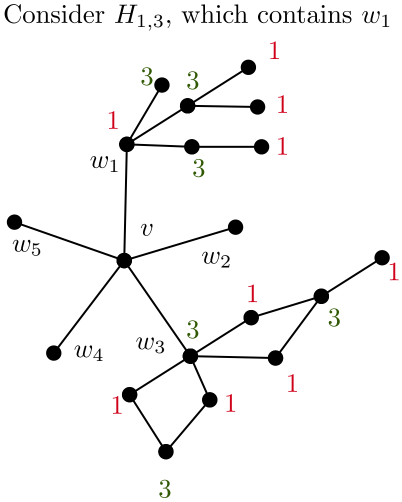
• In the remaining case, every vertex has degree ≥ 5. Let v be a vertex of degree 5.
Embed G properly in the plane. Let neighbours of v in G be w_1, w_2, w_3, w_4, w_5 in clockwise order around.



By inducting $G \setminus v$ has a proper 5-colouring $f : V \setminus \{v\} \to \{1,2,3,4,5\}$. If f only use 4 colours on $\{w_1, w_2, w_3, w_4, w_5\}$, then there is some colour $c \in \{1,2,3,4,5\}$ that we can use for $f(v) = c$ to get a proper 5-colouring of G. By permitting colours, we can say that $f(w_i) \in \{1,2,3,4,5\}$

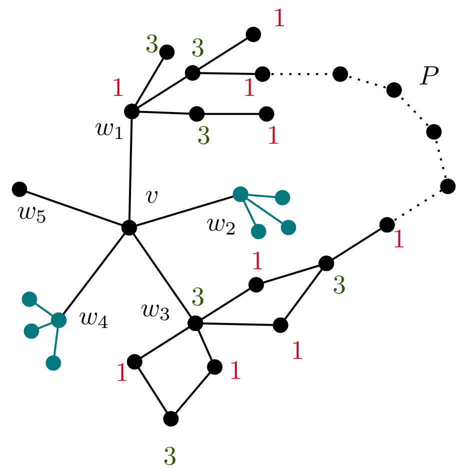
Idea "Rearrange" the colouring $f : V \setminus \{v\} \to \{1,2,3,4,5\}$ to get a new proper 5-colouring g of $G \setminus v$ that uses only 4 colours on the neighbours of v.

For colours $1 \leq i < j \leq 5$, let $H_{i,j}$ be the subgraph of $G \setminus v$ induced by all vertices that are coloured either i or j by f.
Consider $H_{1,3}$, which contains w_1 and w_3.



☺ If w_1 and w_3 are in different components of $H_{1,3}$, then exchange colours $1 \leftrightarrow 3$ on the component of $H_{1,3}$ containing w_1 to get g.
Then let $g(v) = 1$ to get a proper 5-colouring of G.

☹ If w_1 and w_3 are in the same component of $H_{1,3}$, then there is a path P in $H_{1,3}$ form w_1 to w_3, every vertex of which is coloured 1 or 3 by f. Then vertices w_2 and w_4 are on "opposite sides". of the (embedding of the) cycle $C = P \cup \{\{v, w_1\}, \{v, w_3\}\}$.

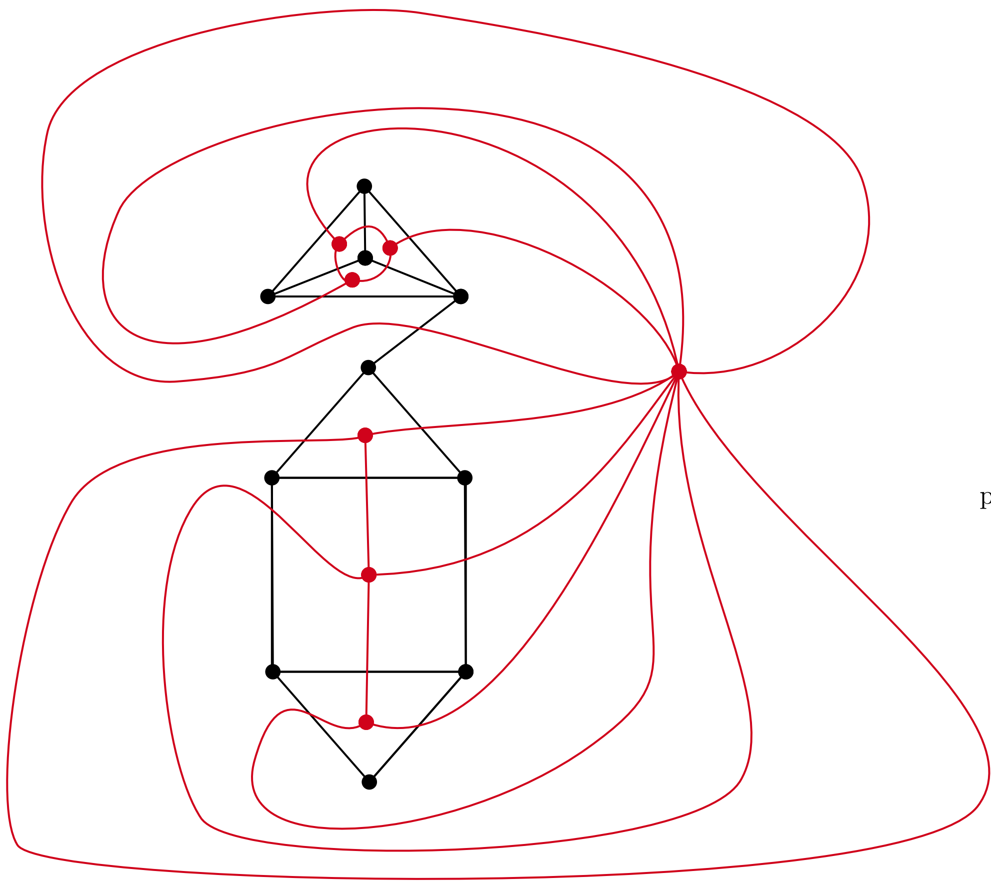


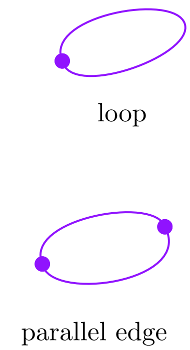
If there is a path Q in $H_{2,4}$ from w_2 to w_4, then it has a vertex z in common with P, by plane embedding. The colour of t is either 1 or 3, but also either 2 or 4. Contradiction. So w_2 and w_4 are in different components of $H_{2,4}$. Back to case ☺. □

9.10 Dual of Planar Graphs

Let $G = (V, E)$ be a graph with plane embedding (P, Γ). The dual of G is G^* (depends on the embedding)

- one vertex of G^* for every face of G
- one edge of G^* going across each edge of G between the corresponding vertices of G^*



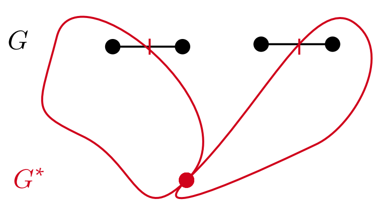


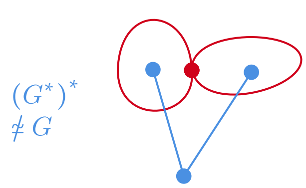
9.10.1 Definition

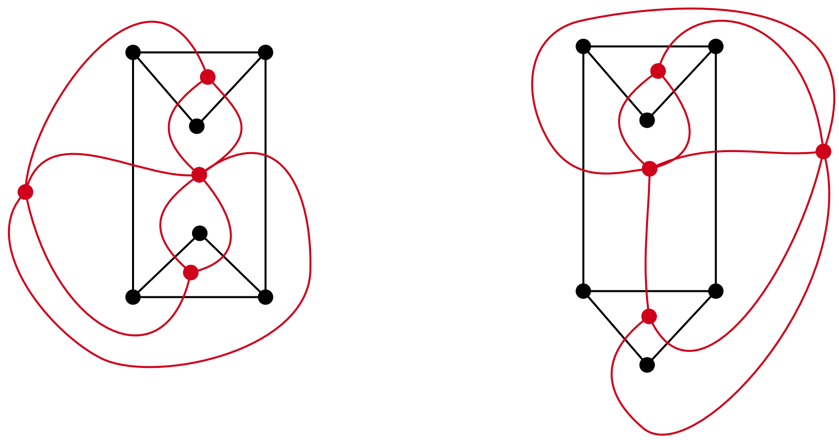
$G = (V, E)$ properly embedded in the plane (P, Γ) set of faces $\mathcal{F}$.
$G^* = (V^*, E^*)$ properly embedded in the plane (P^*, Γ^*)

- one vertex F^* for every face $F \in \mathcal{F}$ of G
- one edge e^* for every edge $e \in E$ of G
- if G is connected then one face v^* for every vertex of G

Eg



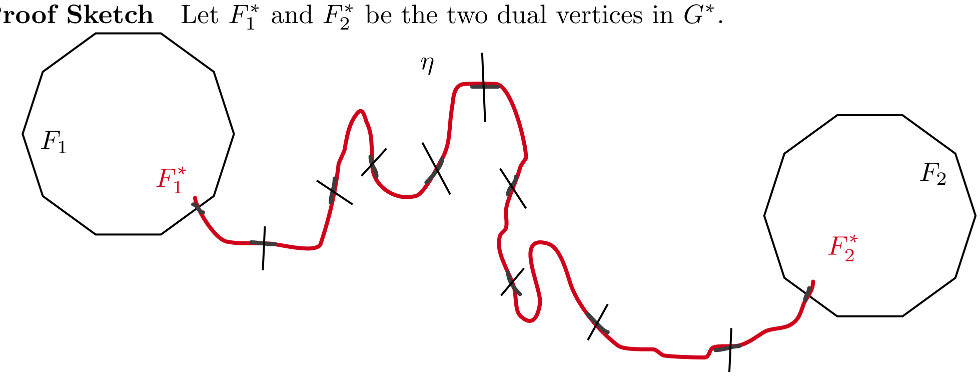




In each cases,
$(G^*)^* \simeq G$
with the original
embedding

Proposition Let $G = (V, E)$ be a plane graph and G^* its dual. Then G^* is connected.

Proof Sketch Let F_1^* and F_2^* be the two dual vertices in G^*.



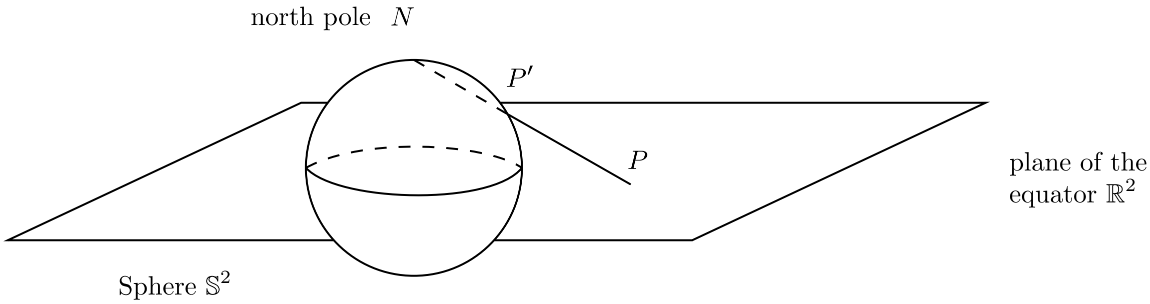
Let η be a continuous curve in $\mathbb{R}^2$ from F_1^* to F_2^*. We can arrange that η intersects each curve γ_e in at most one point and does not pass any points $P_v (v \in V)$.

Look at the sequence of dual faces and dual edges that η induces that is a walk in G^* from F_1^* to F_2^*. $\square$

Proposition If G is a connected plane graph, then $(G^*)^* \simeq G$ with the same embedding.

Lemma If G is connected plane graph and $v \in V$, then the dual edges $\{e^* : v \in e\}$ form the boundary of a face v^* of G^*, and v is the only vertex of G in the interior of G^*.

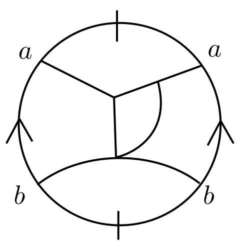
9.11 Graphs on Surfaces



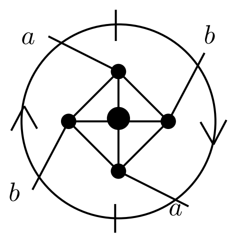
Bijection $\mathbb{R}^2 \to \mathbb{S}^2 \setminus \{N\} : L$. Given $P \in \mathbb{R}^2$, consider the line L through P and N. L intersects the sphere $\mathbb{S}^2$ in a third point P'. The function is $P \mapsto P'$. (stereographic projection).

$$P = P' \iff P \text{ is on the equator of } \mathbb{S}^2$$

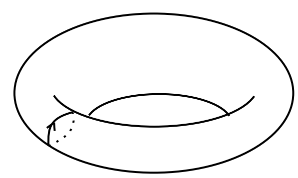
A graph is planar $\iff$ it can be embedded in $\mathbb{S}^2$



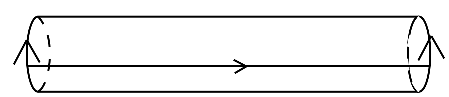
Sphere

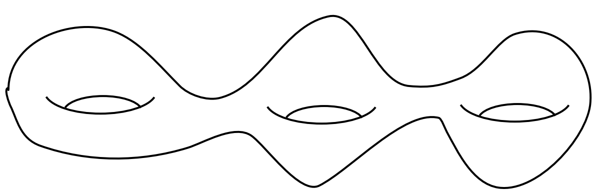


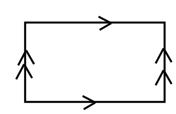
projective plane
with embedded K_5



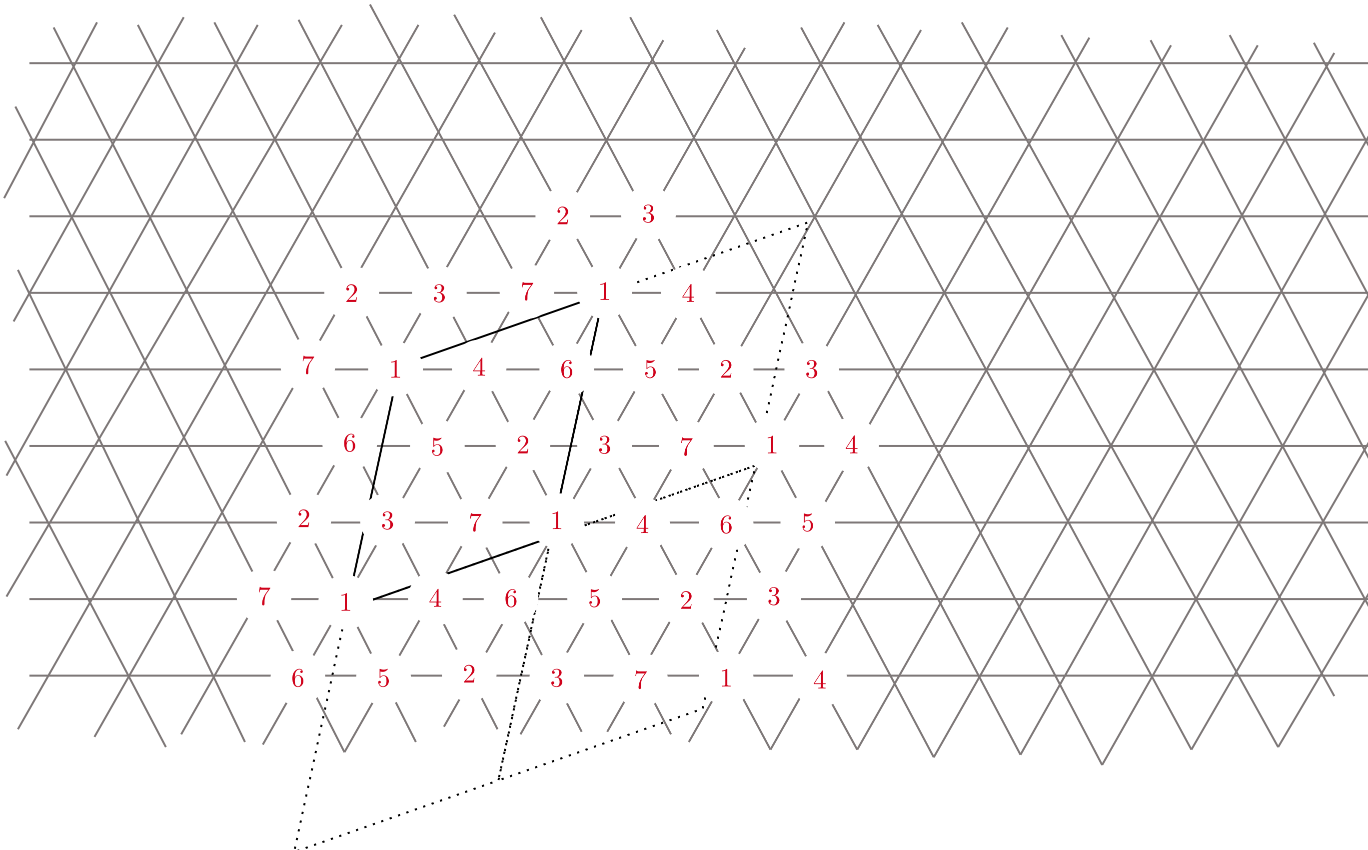
torus

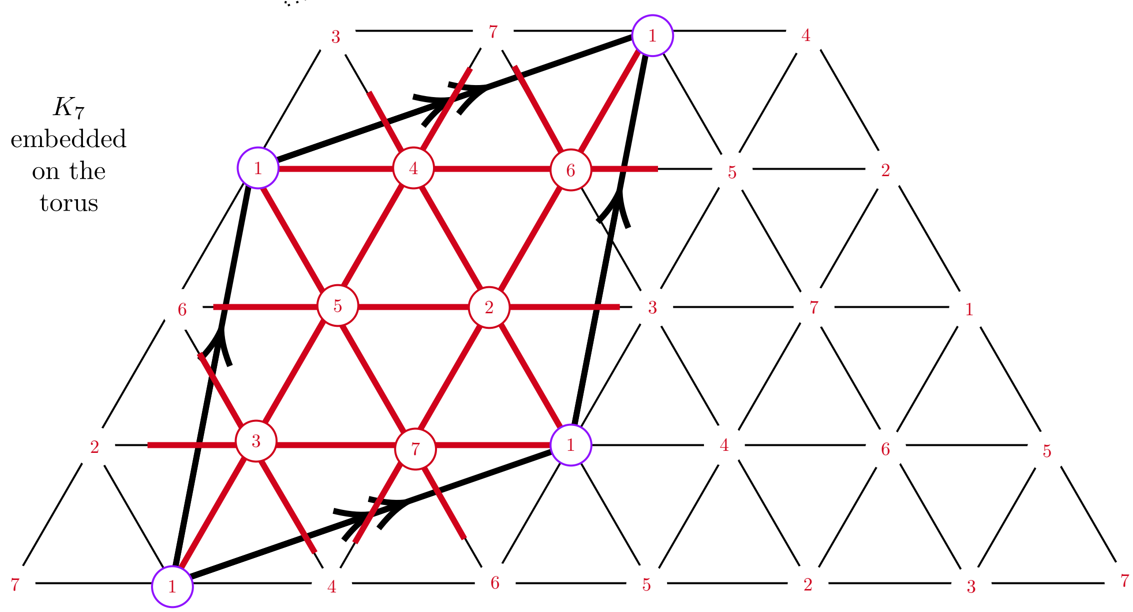






Note In the second figure, a, b just join the vertices together, which means that they are not vertices.





K_7 embedded on the torus

9.12 Heawood conjecture

In graph theory, the Heawood conjecture or Ringel-Youngs theorem gives a lower bound for the number of colors that are necessary for graph coloring on a surface of a given genus. For surfaces of genus 0, 1, 2, 3, 4, 5, 6, 7, ..., the required number of colors is 4, 7, 8, 9, 10, 11, 12, 12, ... the chromatic number or

Heawood number. [3]

Statement Percy John Heawood conjectured in 1890 that for a given genus g ¿ 0, the minimum number of colors necessary to color all graphs drawn on an orientable surface of that genus (or equivalently to color the regions of any partition of the surface into simply connected regions) is given by

$$\gamma(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor$$

Replacing the genus by the Euler characteristic, we obtain a formula that covers both the orientable and non-orientable cases,

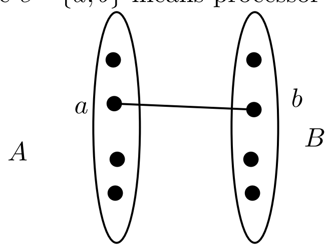
$$\gamma(\chi) = \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor.$$

[3] all from wikipedia...

10 | Matching Theory

10.1 Job Assignment Problem

Graph $G = (V, E)$ with bipartition $U = A \cup B$.
A: processors B: jobs
edge $e = \{a, b\}$ means processor a can do job b.

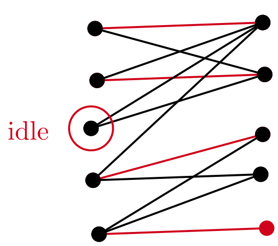


Problem Assign jobs to processors so that the number of working processors is maximized.

Constraints

- each job can be assigned to at most one processor
- each processor can be assigned at most one job
- each job is assigned to a processor that can do it

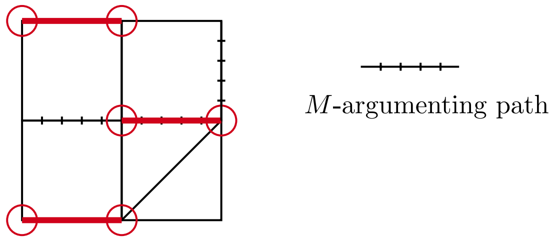
eg//



10.1.1 Matchings

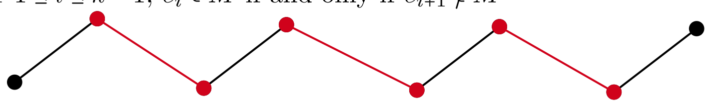
Given a graph $G = (V, E)$. A matching in G is a subset $M \subseteq E$ of edges such that in (V, M) every vertex

has degree	either 0	unsaturated	(by M)
	or 1	saturated	(by M)



10.1.2 M-alternating path

$P : v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$ a path in G.
For $1 \le i \le k-1$, $e_i \in M$ if and only if $e_{i+1} \notin M$



10.1.3 M-augmenting path

An M-alternating path with ≥ 1 edge & both end vertices of M are unsaturated.

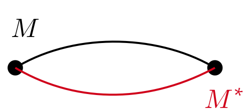
10.1.4 "Flipping" an alternating path

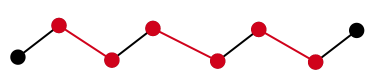
M a matching
P a M-alternating path
$M' = (M \setminus (M \cap E(P)) \cup (E(P) \setminus M))$

Note M' might not be a matching.
If P is M-augmenting, then M' is a matching and $|M'| = 1 + |M|$ $\quad (*)$

Proposition Let $G = (V, E)$ be a graph and $M \subseteq E$ a matching of G. M is as big as possible if and only if M does not have an augmenting path.

Proof If M is maximum size, then it does not have an augumenting path by $(*)$.
Suppose M, M^* [1] are matchings with $|M| < |M^*|$. So M is not maximize size. Some component of $(V, M \cup M^*)$ has strictly more M^*-edges than M-edges.
Look at the components of $(V, M \cup M^*)$

- single vertex
- an edge in $M \cap M^*$ 

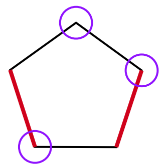
- even cycle 
- alternating path for both M and M^* 

Since $|M^*| > |M|$ there is an alternating path in $(V, M \cup M^*)$ with both ends in M^*. This is an M-augmenting path. $\square$

[1] not related to "dual"...

10.2 Covers

Given a graph $G = (V, E)$. A cover is a subset $C \subseteq V$ such that every edge in E is incident with at least one vertex in C. We want to minimize $|C|$.



$\min |C| = 3$ $\max |M| = 2$

Proposition Let $G = (V, E)$ be a graph, M a matching, and C a cover. Then $|M| \le |C|$.

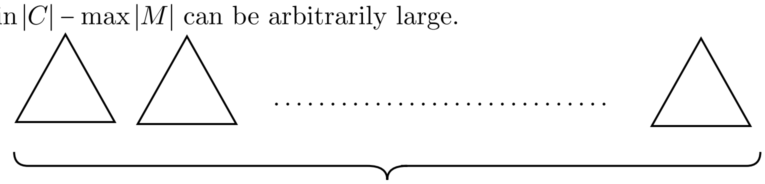
Proof Let $X = \{(v, e) : v \in C,\ e \in M,\ \text{ and } v \in e\}$

$$|X| = \sum_{v \in C} \sum_{\substack{e \in M \\ e \ni v}} 1 \le \sum_{v \in C} 1 = |C|$$

$$|X| = \sum_{e \in M} \sum_{\substack{v \in C \\ v \in e}} 1 \ge \sum_{e \in M} 1 = |M|$$

So $|M| \le |X| \le |C|$. $\square$

There can be a "gap" between $\max |M|$ & $\min |C|$.
$\min |C| - \max |M|$ can be arbitrarily large.



k copies

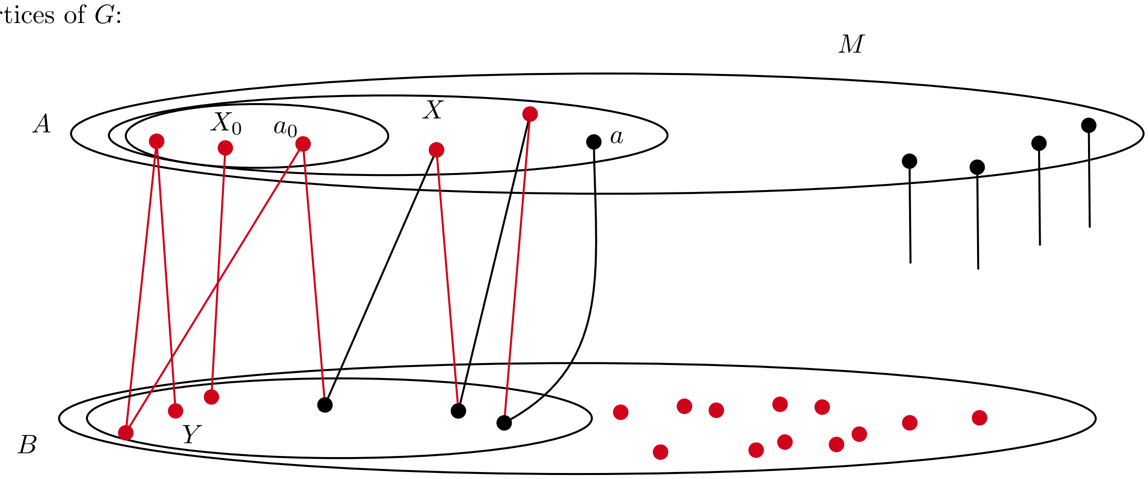
Try to consider other examples that are connected. Then we have a following theorem.

10.3 Kőnig's Theorem

If G is bipartite, then $\max |M| = \min |C|$.

10.3.1 Anatomy of a bipartite matching

Let $G = (V, E)$ be a graph with bipartition (A, B). Let M be a matching in G. Define several sets of vertices of G:



$X_0 = \{a \in A :\ a \text{ is not saturated by } M\}$
Every vertex in $A \setminus X_0$ is M-saturated

$X = \{a \in A :\ a \text{ is joined to some } a_0 \in X_0 \text{ by } M\text{-alternating path}\}$
The edge of this path with a is in M.

$Y = \{b \in B :\ b \text{ is joined to some } a_0 \in X_0 \text{ by } M\text{-alternating path}\}$

Lemma 1 If $b \in Y$ is not M-saturated, then M has an augmenting path, so M is not maximum. [2]

Lemma 2 Edges between X and $B \setminus Y$ do not exist.

Proof Let $a \in X$, $a_0 e_1 v_1 \dots v_{k-1} e_k a_k = a$ be an M-alternating path from $a_0 \in X_0$ to $a \in X$. Consider and edge $f = \{a, b\}$ from a to $b \in B$. If $f \in M$, then $f = e_k$, so $b = v_{k-1} \in Y$, so $b \notin B \setminus Y$. If $f \notin M$, then $a_0 e_1 v - 1 \dots v_{k-1} e_k a_k f b$ is M-alternating, so $b \in Y$, so $b \notin B \setminus Y$. □

Lemma 3 There are no edges of M between Y and $A \setminus X$.

Proof If there is a matching edge from $b \in Y$ to $a \in A$, then there is an M-alternating path from some $a_0 \in X_0$, to b, to a, so $a \in X$, so $a \notin A \setminus X$. □

Note By Lemma 2, $C = Y \cup (A \setminus X)$ is a cover.

10.3.2 Proof for Kőnig's Theorem

Adopt all above notation. $G = (V, E)$ with bipartition (A, B), matching M, sets X_0, X, Y.

Assume that M is maximum matching. By Lemma 1, every $b \in Y$ is M-saturated. By Lemma 3, the edges of M incident with vertices in Y give a bijection between Y and $X \setminus X_0$, so $|Y| = |X \setminus X_0|$.

Every vertex in $A \setminus X$ is M-saturated by a matching edge with its other end in $B \setminus Y$. (by Lemma 3)

matching M has size

$$|M| = |A \setminus X_0| = |A \setminus X| + |X \setminus X_0| = |A \setminus X| + |Y| = |C|$$

Therefore, we are done! □

10.4 A Bipartite Matching Theorem

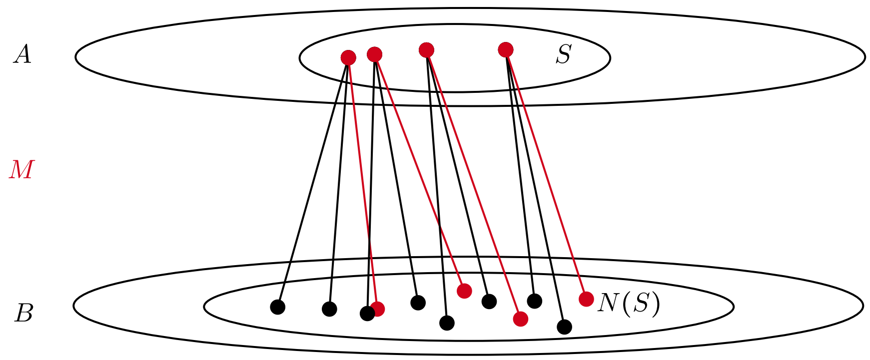
Missing...

10.5 Hall's Theorem

10.5.1 A-saturated matchings

A matching in a graph with bipartition (A, B) such that every vertex in A is saturated.

[2] proof is left as an exercise...



For $S \subseteq A$, let the neighbourhood of S be $N(S) = \{b \in B : \{a,b\} \in E \quad \text{for some } a \in S\}$. M gives an injection from S to $N(S)$.

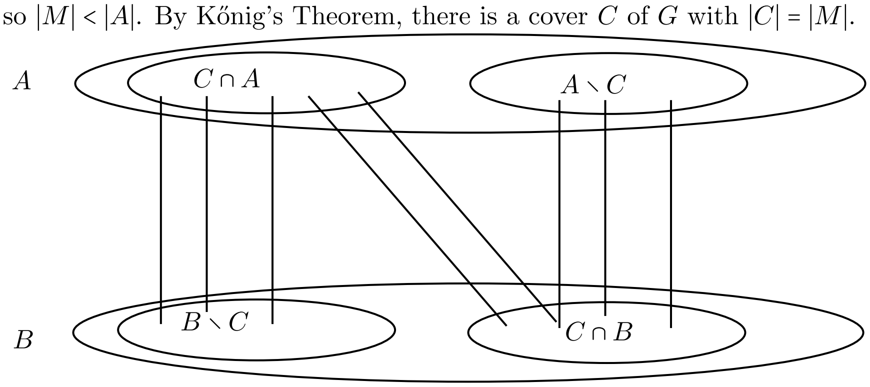
Necessary Condition (Hall's Condition):
If G has an A-saturated matching, then $\forall S \subseteq A : |S| \leq |N(S)|$.

10.5.2 Statement

Let $G = (V,E)$ has bipartition (A,B), then G has an A-saturating if and only if Hall's condition holds.

10.5.3 Proof

If G has an A-saturated matching, then Hall's condition holds. If not, let M be a maximum matching in G: so $|M| < |A|$. By Kőnig's Theorem, there is a cover C of G with $|C| = |M|$.



Let $S = A \setminus C$.

$$|A| - |A \cap C| = |A \setminus C| = |S|$$

$$|N(S)| \leq |C \cap B| = |C| - |A \cap C| = |M| - |A \cap C| < |A| - |A \cap C|$$

So $|N(S)| < |S|$. So Hall's condition fails. □

$(*)$ Let G be k-regular with bipartition (A,B), then $|A| = |B|$. (if $k \geq 1$)

Proof $k|A| = |E| = k|B|$. Divide by k. □

10.5.4 Corollary to Hall's Theorem

Let $G = (V,E)$ be k-regular with bipartition (A,B), $k \geq 1$, then G has a perfect matching.

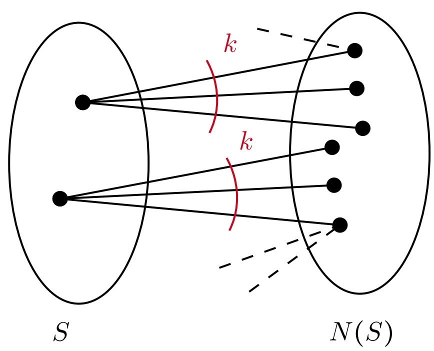
Proof We want to find an A-saturating matching, which will be perfect, by $(*)$. We will verify Hall's condition.

Consider any $S \subseteq A$. Let $N(S) = \{b \in B : \{a, b\} \in E \text{ for some } a \in S\}$.

The number of edges with a vertex in S is $k \cdot |S|$.

The number of edges incident with a vertex in $N(S)$ is $k \cdot |N(S)|$.

Define $\partial F = \{e \in E : e$ is incident with some vertex in $F\}$ = "boundary" of F. [3]



Note that $\partial S \leq \partial N(S)$.
So

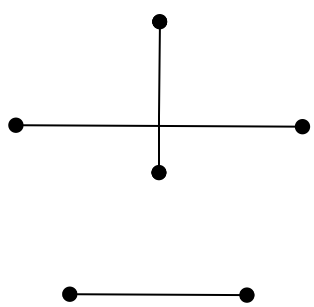
$$k|S| = |\partial S| \leq |\partial N(S)| = kN(S)$$

Divide by k. □.

Corollary If G is k-regular and bipartite, then $E(G)$ can be partitioned into k pairwise disjoint perfect matchings.

eg

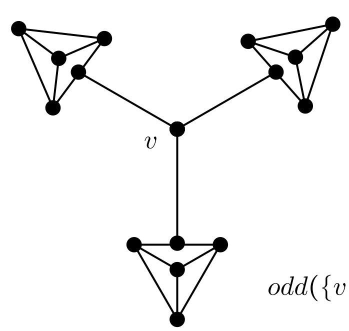
K_6



5 rotated copies
partition $E(K_6)$

10.5.5 Regular Graphs with No Perfect Matching

- $k = 2$: odd cycles
- $k = 3$:



no perfect matching

$odd(\{v\}) = 3$

[3]not a formal definition

Let $G = (V, E)$ be a graph. For $S \subseteq V$, let $odd(S)$ be the number of connected components of $G \setminus S$ that have an odd number of vertices.

If G has a perfect matching, then for all $S \subseteq V$, $odd(S) \leq |S|$.

10.6 Tutte's Theorem

A graph G has a perfect matching if and only if $\forall S \subseteq V$, $odd(S) \leq |S|$.

11 | Final Exam

150 mins 8 questions
55 points total
12 or 13 points enumeration
43 or 42 points graph theory

11.1 Enumeration

- power series
- recurrence relations
- binomial series

$$(t \geq 1): \frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

- partial fractions

- generating functions
- compositions
- binary strings
- compute averages

$$\frac{\partial}{\partial y} \Phi(x,y)\bigg|_{y=1} \qquad \text{etc.}$$

Transition Matrix and block pattern will not be tested...

11.2 Graph Theory

- basic definitions: read through the notes and rewrite them in a paper
- handshake lemma & bipartite version
- isomorphism (finding the cycles might be useful)
- walks paths, cycles, bipartiteness
- connectedness, cut-edges, trees
- spanning trees, search trees
- planar graphs
  - (definitions) embedding

  - handshake lemma for faces, cut-edges & faces
  - Euler's formula
  - K_5 and $K_{3,3}$ are not planar subdivision, Kuratowski's Theorem
  - numerology $\quad q \leq 3p - 6. \quad \exists$ vertex of degree ≤ 5.

- colouring graphs
  - chromatic numer $\chi^{(G)}$
  - six-colour theorem (& proof)
  - five-colour theorem
  - four-colour theorem

  Non-planar graph can also be coloured

- Matching
  - definitions
  - Lemma: M maximum $\iff$ no augmenting path
  - $\max|M| \leq \min|C|$, possible gap
  - Kőnig's Theorem (with proof (3 lemmas))
    We are trusted on this algorithm, so it will not be tested...
    (Anatomy of a bipartite matching)
  - Hall's Theorem