245e notes

Sibelius Peng

April 14, 2019

Abstract

This notes contains the lecture notes since lec 09

Contents

Chapter 1

lec 09

syntax	semantics
Define wff	truth tables, satisfiability

1.1 A proof system for propositional logic

Desirable properties of proof system:

1. It can only prove "correct" statements
2. There is a "method" to check a given object is a valid proof
3. Every "correct" statement has a proof

When "correct statement" stands for a propositional tautology, a proof of α will be a truth table for α that has only T's in the rightmost column.

Our proof system consists of two components: Axioms and Deduction rules.

Axioms The set of axioms - and wff of one of the following forms:

1. $(\alpha \to (\beta \to \alpha))$
2. $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$
3. $(((\neg\alpha) \to (\neg\beta)) \to (\beta \to \alpha))$

Example

$$(\underbrace{((\neg p) \to q)}_{\alpha} \to (\underbrace{p}_{\beta} \to \underbrace{((\neg) p \to q)}_{\alpha}))$$

It's an axiom.

$$((p \to q) \to ((\neg q) \to (\neg p)))$$

Not an axiom. purely syntactic. check it symbol by symbol.

$\{\neg, \to\}$ is an adequate set of connectives.

Examples $(p \to p)$ is not one of axioms (but a tautology), just 5 symbols -every axiom has more.

Deduction Rule The second component of the proof system is a deduction rule. $\frac{\alpha, \alpha\to\beta}{\beta}$. Modus ponens.

The set of provable wff's is

$$I(Axioms,\ \{modus\ ponens\})$$

- A formal proof is a construction sequence in the structure. Namely, $\alpha_1, \ldots, \alpha_m$ is a formal proof of some β, if each α_i is either an axiom or the result of applying Modus Ponens (MP) to some α_j, α_k for $j, k < i$.

Example
$(\alpha \to ((\alpha \to \alpha) \to \alpha))$ axiom 1
$((\underbrace{\alpha}_{A} \to (\underbrace{(\alpha \to \alpha)}_{B} \to \underbrace{\alpha}_{C})) \to ((\underbrace{\alpha}_{A} \to \underbrace{(\alpha \to \alpha)}_{B}) \to (\underbrace{\alpha}_{A} \to \underbrace{\alpha}_{C})))$ Ax 2
Then MP, $((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$
$(\alpha \to (\alpha \to \alpha))$ Ax 1
MP, $\alpha \to \alpha$.

Notation We will use $\vdash$ to denote "formally proves". $\vdash \alpha$. " α has a formal proof."

Example For every α, $\vdash ((\neg(\neg\alpha)) \to \alpha)$

$$\begin{cases} (\neg\neg\neg\alpha \to (\neg\alpha \to \neg\neg\neg\alpha)) & Ax1 \\ ((\neg\neg\neg\alpha \to (\neg\alpha \to \neg\neg\neg\alpha)) \to ((\neg\neg\neg\alpha \to \neg\alpha) \to (\neg\neg\neg\alpha \to \neg\neg\neg\alpha))) & Ax2 \\ ((\neg\neg\neg\alpha \to \neg\alpha) \to (\neg\neg\neg\alpha \to \neg\neg\neg\alpha)) & MP \\ (((\neg\neg\neg\alpha \to \neg\alpha) \to (\neg\neg\neg\alpha \to \neg\neg\neg\alpha)) \to (next\ time)) & \end{cases}$$

Theorem (Soundness) For every wff α, if $\vdash \alpha$ then α is a tautology.

Proof By generalized induction on $I(Axioms,\ MP)$. Base case: Check that every axiom is tautology. Induction Step: Assume both α and $(\alpha \to \beta)$ are tautologies, need to show that β is a tautology.
By way of contradiction, assume β is not a tautology, then for some truth assignment v, $\overline{v}(\beta) = F$. $\overline{v}(\alpha) = T$ since α is a tautology by IH. $\overline{v}(\alpha \to \beta) = F$ contradicts the assumption "$(\alpha \to \beta)$ is tautology". □

$\vdash \alpha$ (syntax). α tautology (semantics)

Chapter 2

L10

2.1 Formal Proofs in Prop Logic

The set of formal theorems is

$$Th = I(Axioms,\ \{MP\})$$

A formal proof is a construction sequence for this structure.
We use $\vdash \alpha$ to denote "α is a formal theorem".

We showed that, for any α, $\vdash (\alpha \to \alpha)$.

The Soundness Theorem Any formal theorem is a tautology.

2.2 Proof from assumptions

Let Σ be a set of wff's. The set of formal theorems under the assumptions Σ is

$$Th(\Sigma) = I(Axioms \cup \Sigma, \{MP\})$$

$\alpha_1, \ldots, \alpha_n$ is a formal proof under Σ if each α_i is either axiom or a member of Σ or follows from some $\alpha_j, \alpha_k, j, k < i$ using MP.

Example Let $\Sigma = \{A, (A \to B)\}$. Claim: $\{A, (A \to B)\} \vdash B$
A $(\in)$
$(A \to B)$ $(\in)$
B MP

Example For every α, β, $\{\alpha, (\neg\alpha)\} \vdash \beta$

	$((\neg\alpha) \to ((\neg\beta) \to (\neg\alpha)))$	$Ax1$
	$(\neg\alpha)$	$\in$
	$((\neg\beta) \to (\neg\alpha))$	MP
Formal Proof	$(((\neg\beta) \to (\neg\alpha)) \to (\alpha \to \beta))$	$Ax3$
	$(\alpha \to \beta)$	MP
	α	$\in$
	β	MP

2.3 Properties of proofs from assumptions

Extended Soundness Theorem For every Σ, α, if $\Sigma \vdash \alpha$, then $\Sigma \vDash \alpha$. (Substituting $\Sigma = \emptyset$, yields the basic Soundness Theorem)

Proof (Notice that we can rephrase this theorem as: for any $\alpha \in I(Ax \cup \Sigma, \{MP\}), \Sigma \vDash \alpha$)
I leave the proof to the assignment.

Monotonicity For every Σ, Σ', α. If $\Sigma \vdash \alpha$, and $\Sigma \subseteq \Sigma'$, then $\Sigma' \vdash \alpha$. In other words, $\Sigma \subseteq \Sigma' \implies Th(\Sigma) \subseteq Th(\Sigma')$.
(Note that human reasoning does not enjoy this property)

Quest $\vdash^{(hr)}$ There will be a class today
Quest, Piazza (Shai is sick) $\nvdash^{(hr)}$ There will be a class today

Proof Since $\Sigma \vdash \alpha$, thereis a construction seq (formal proof) of αm based on $\Sigma \cup Ax$,

$$\alpha_1 \ldots \alpha_n = \alpha$$

each α_1 is either an axiom or $\alpha_i \in \Sigma$, but in that case, $\alpha_1 \in \Sigma'$, or α_i is the outcome of MP on earlier α_j, α_k, therefore $\alpha_1 \ldots \alpha_n$ is also a proof from Σ'.

Strong Monotonicity If for every $\alpha \in \Sigma', \Sigma \vdash \alpha$, then $Th(\Sigma') \subseteq Th(\Sigma)$.

Example Let $\Sigma = \{A, (A \to B)\}.\Sigma' = \{B\}$.
$\Sigma \vdash \Sigma'$ (every $\alpha \in \Sigma'$ has a proof under Σ) $\implies Th(\{B\}) \subseteq Th(\{A, (A \to B)\})$

The deduction theorem For every Σ, α, β, $\quad \Sigma \cup \{\alpha\} \vdash \beta$ if and only if $\Sigma \vdash (\alpha \to \beta)$
(in particular $\{\alpha\} \vdash \beta$ iff $\vdash (\alpha \to \beta)$)

Example For every α, β, γ. $\quad \{(\alpha \to \beta, \beta \to \gamma)\} \vdash (\alpha \to \gamma)$

Proof Applying deduction theorem, if suffices to show

$$\{(\alpha \to \beta, \beta \to \gamma), \alpha\} \vdash \gamma$$

Proof of deduction theorem
Easy direction Assume $\Sigma \vdash (\alpha \to \beta)$, by monotonicity,
$\Sigma \cup \{\alpha\} \vdash (\alpha \to \beta)$
by strong monotonicity, it suffices to show that

$$\Sigma' = \Sigma \cup \{\alpha\} \cup \{\alpha \to \beta\} \vdash \beta$$

$\Sigma \vdash \Sigma'$
We know $\alpha, \alpha \to \beta, \beta(MP)$ is a proof from Σ'

The harder direction Show that if $\Sigma \cup \{\alpha\} \vdash \beta$ then $\Sigma \vdash (\alpha \to \beta)$.
Rephrase this statement as, for all $\beta \in Th(\Sigma \cup \{\alpha\}), \Sigma \vdash (\alpha \to \beta)$.
Now prove by generalized induction on $Th(\Sigma \cup \{\Sigma \cup \{\alpha\}\}) = I(Ax \cup \Sigma \cup \{\alpha\}, MP)$

Base case β is in the core set

- case 1: β is an axiom
$$\begin{array}{ll} \beta & (axiom) \\ \beta \to (\alpha \to \beta) & ax1 \\ (\alpha \to \beta) & MP \end{array}$$

- case 2: $\beta \in \Sigma$
$$\begin{array}{ll} \beta & assumption \\ \beta \to (\alpha \to \beta) & ax1 \\ \alpha \to \beta & MP \end{array}$$

- case 3: $\beta = \alpha$.
  So $\Sigma \vdash (\alpha \to \beta)$ is, in fact, $\Sigma \vdash (\alpha \to \alpha)$, we showed that $\emptyset \vdash (\alpha \to \alpha)$. Use monotonicity.

The induction Step

$\frac{\gamma, (\gamma \to \delta)}{\delta}$ MP
Assume $\Sigma \vdash (\alpha \to \gamma)$ and $\Sigma \vdash (\alpha \to (\gamma \to \delta))$ need to show $\Sigma \vdash (\alpha \to \delta)$
So by our assumptions plus strong monotonicity, imply that it suffices to show:

$$\Sigma \cup \{(\alpha \to \gamma), (\alpha \to (\gamma \to \delta))\} \vdash (\alpha \to \delta)$$

$$\begin{array}{ll} (\alpha \to (\gamma \to \delta)) \to ((\alpha \to \gamma) \to (\alpha \to \delta)) & \text{Ax2} \\ (\alpha \to (\gamma \to \delta)) & \text{ass} \\ (\alpha \to \gamma) \to (\alpha \to \delta) & \text{MP} \\ (\alpha \to \gamma) & \text{Ass} \\ (\alpha \to \delta) & \text{MP} \end{array}$$

Example Claim: for every $\alpha, \vdash (\neg\neg\alpha \to \alpha)$.

Proof By the deduction theorem, it suffices to show $\{(\neg\neg\alpha)\} \vdash \alpha$

formal proof
$$\begin{array}{ll} ((\neg\neg\alpha) \to (\neg\neg\neg\neg\alpha \to \neg\alpha)) & \text{Ax1} \\ \neg\neg\alpha & \text{Assumption} \\ (\neg\neg\neg\neg\alpha \to \neg\neg\alpha) & \text{MP} \\ (\neg\neg\neg\neg\alpha \to \neg\neg\alpha) \to (\neg\alpha \to \neg\neg\neg\alpha) & \text{Ax3} \\ (\neg\alpha \to \neg\neg\neg\alpha) & \text{MP} \\ ((\neg\alpha \to \neg\neg\neg\alpha) \to (\neg\neg\alpha \to \alpha)) & \text{Ax3} \\ \neg\neg\alpha \to \alpha & \text{MP} \\ \neg\neg\alpha & \text{ass} \\ \alpha & \text{MP} \end{array}$$

2.4 Important syntactic notion - Consistency

Definition 1 Σ is consistent if

For no $\alpha, \Sigma \vdash \alpha$ and $\Sigma \vdash (\neg\alpha)$.

Definition 2 Σ is consistent if there exists some α such that $\Sigma \nvdash \alpha$.

Claim The two definitions are equivalent.

Chapter 3

L11

Definition 1 A set of wff's Σ is consistent if

For no $\alpha, \Sigma \vdash \alpha$ and $\Sigma \vdash (\neg\alpha)$.

Definition 2 Σ is consistent if there exists some α such that $\Sigma \nvdash \alpha$.

Claim The two definitions are equivalent.

Proof Assume that for no α does $\Sigma \vdash \alpha$, and $\Sigma \nvdash \alpha$.
Pick any α, say $\alpha \equiv p$, either $\Sigma \nvdash \alpha$, or, if it does, then $\Sigma \nvdash \neg\alpha$. In any case, we found some β such that $\Sigma \nvdash \beta$.

Assume that for some $\beta, \Sigma \nvdash \beta$. If the first definition is violated, then for some α, $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$. Therefore $\Sigma \vdash \Sigma \cup \{\alpha, \neg\alpha\} \vdash \{\alpha, \neg\alpha\}$
$\{\alpha, \neg\alpha\} \vdash \beta$ for every α, β. So $\Sigma \vdash \beta$ for every β, contradiction.

Let $\Sigma \subseteq \Sigma'$ if Σ' is consistent, then so is Σ.

Corollary If any set of wff's is consistent, then in particular, $\emptyset$ is consistent.

Is $\emptyset$ consistent?

The soundness Theorem If $\vdash \alpha$, then α is tautology. In particular, $\nvdash p$.

syntax	semantics
definition of wff	truth assignments
proof system	α satisfiable
Σ consistent	Σ satisfiable
$\vdash \alpha$	α tautology
	$\alpha \vDash \beta$
$\Sigma \vdash \alpha$	$\Sigma \vDash \alpha$
	compactness theorem

Theorem Every satisfiable Σ is consistent.

Proof By way of contradiction, assume Σ is inconsistent. So for some α, both $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$. If Σ is satisfiable, then for some truth assignment v, v satisfies all wff's in Σ. By soundness, $\Sigma \models \alpha$ and $\Sigma \models \neg\alpha$. So for that assignment v, we get $v(\alpha) = T$, and $v(\neg\alpha) = T$, violating the truth table of $\neg$. Contradiction.

Is $\Sigma = \{(p_i \to p_j) : i, j \in \mathbb{N}\}$ consistent? By out previous theorem, it suffices to show that Σ is satisfiable. For example that all-true v satisfies every member of Σ.

Definition We say that Σ is maximally consistent if Σ is consistent but, for every α either $\Sigma \vdash \alpha$ or $\Sigma \cup \{\alpha\}$ is inconsistent.
(note that whenever $\Sigma \vdash \alpha$, $\Sigma \cup \{\alpha\}$ is no stronger than Σ. Namely, for every β, if $\Sigma \cup \{\alpha\} \vdash \beta$, then $\Sigma \vdash \beta$.)

Example Let $\Sigma \equiv \{p_1\}$ (over the variables $p_1 p_2 \ldots$)

Claim 1 $\{p_1\}$ is consistent. Since it is satisfiable.

Claim 2 $\{p_1\}$ is not maximally consistent.
Let $\alpha = p_3$. $\{p_1\} \nvdash p_3$, by soundness, it suffices to show that $\{p_1\} \not\models p_3$. Consider any truth assignment such that $v(p_1) = T, v(p_3) = F$, then v satisfies p_1 but not p_3.

Finally note that $\{p_1, p_3\}$ is consistent, since it is satisfiable.

Claim $\Sigma\{p_i : i \in \mathbb{N}\}$ is maximally consistent (over $\{p_1 p_2 \ldots\}$)

Proof Why is Σ consistent? It is satisfiable by all T assignment.

Why is it maximally consistent? Need to show that for every α, if $\Sigma \nvdash \alpha$, $\Sigma \cup \{\alpha\}$ is inconsistent. We can already show that if $\Sigma \not\models$, then $\Sigma \cup \{\alpha\}$ is not satisfiable. (since Σ is maximally satisfiable)

Σ consistent $\sim$ A set of vectors A is linearly independent

Σ maximally consistent $\sim$ A set of vectors A is maximally linearly independent $\sim$ A is a basis

Lemma For every consistent Σ, there exists a maximally consistent $\Sigma' \supseteq \Sigma$

Proof Let $\alpha_1 \alpha_2 \ldots \alpha_n \ldots$ be a list of all wffs over $\{p_1 \ldots p_n \ldots\}$
Let Σ_0 be Σ, and construct a sequence of sets of wffs

$$\Sigma_0 \subseteq \Sigma_1 \ldots \subseteq \Sigma_n \subseteq \ldots$$

such that

1. each Σ_i is consistent.
2. For every i, either $\Sigma_i \vdash \alpha_i$ or $\Sigma_i \nvdash \alpha_i$.

The construction of the Σ_i's is by induction on i.

$\Sigma_0 = \Sigma$ clearly satisfies our requirements 1 and 2.

Given Σ_i if $\Sigma_i \vdash \neg\alpha_i$, let $\Sigma_{i+1} = \Sigma_i$. (then if Σ_i satisfies 1 and 2, then so will Σ_{i+1}).

If on the other hand, $\Sigma_i \nvdash \neg\alpha_{i+1}$, then let $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$

Claim If Σ_i satisfies 1 and 2, then so does Σ_{i+1}.

Proof of claim requirement 2 follows by the definition of Σ_{i+1}. But why is Σ_{i+1} consistent? By way of contradiction, otherwise $\Sigma_{i+1} \vdash \neg\alpha_{i+1}$ (inconsistent Σ proves everything)

Claim if $\Sigma \cup \{\alpha\} \vdash \neg\alpha$, then $\Sigma \vdash \neg\alpha$ (for every Σ, α)

by the claim, if Σ_{i+1} is inconsistent, the $\Sigma_{i+1} \vdash \neg\alpha_{i+1}$ but $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$ so $\Sigma_i \vdash \neg\alpha_{i+1}$. Contradiction.

We assumed $\Sigma_i \nvdash \neg\alpha_{i+1}$ and defined $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$.

Finally define

$$\Sigma' = \bigcup_{i \in \mathbb{N}} \Sigma_i$$

is maximally consistent.

Chapter 4

L12

Thm Any consistent set of wffs Σ can be extended $\Sigma' \supseteq \Sigma$ that is maximally consistent.

Proof We construct a sequence of sets of wffs $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \Sigma_i \subseteq \Sigma_{i+1} \ldots$
s.t

1. $\Sigma_0 = \Sigma$
2. For all i, Σ_i is consistent
3. For some fixed enumeration of all wff's. $\alpha_1, \alpha_2, \ldots \alpha_n \ldots$ (over $p_1, \ldots p_n$)
   For all i, either $\Sigma_i \vdash \alpha_i$ or $\Sigma_i \vdash \neg\alpha_i$

   Assuming Σ_i is defined and meets the requirements. Let $\Sigma_{i+1} = \begin{cases} \Sigma_i & \text{if } \Sigma_i \vdash \neg\alpha_{i+1} \\ \Sigma_i \cup \{\alpha_{i+1}\} & \text{otherwise} \end{cases}$

Claim Assuming Σ_i meets the requirements then so will Σ_{i+1} (defined above)

Side Claim For every Σ, α, if $\Sigma \cup \{\alpha\} \vdash \neg\alpha$, then $\Sigma \vdash \neg\alpha$

To prove the side claim, it suffices to show that

$$\vdash (\alpha \to \neg\alpha) \to \neg\alpha$$

why does it suffice? Assume $\Sigma \cup \{\alpha\} \vdash \neg\alpha$. Then by deduction thm, $\Sigma \vdash (\alpha \to \neg\alpha)$

Corollary Our construction of Σ_i's can be carried out while respecting requirements 1,2,3

Given Σ, we constructed $\Sigma \subseteq \ldots \Sigma_i \subseteq \ldots$
Now define

$$\Sigma' = \bigcup_{i=1}^{\infty} \Sigma_i$$

Claim The Σ' we constructed is maximally consistent.

Proof maximality: For every α, $\alpha = \alpha_i$ for some $i \in \mathbb{N}$. Therefore by 2, $\Sigma \vdash \alpha_i$ or $\Sigma \vdash \neg\alpha_i$. In other words, $\Sigma_i \vdash \alpha$ or $\Sigma_i \vdash \neg\alpha$.
Each Σ_i is a subset of $\Sigma'(= \cup\Sigma_i)$, therefore, by monotinicity, $\Sigma' \vdash \alpha$ or $\Sigma' \vdash \neg\alpha$.

Consistency: by contradiction, it's not. In that case, for some α, $\Sigma' \vdash \alpha$ and $\Sigma' \vdash \neg\alpha$. Let $\beta_1 \dots \beta_k$ be a formal proof of α from Σ'. Let $\gamma_1 \dots \gamma_k$ be a formal proof of $\neg\alpha$ from Σ'.

each β_i that is an assumption from Σ' belongs to some Σ_{m_i}. Similarly, each γ_i that is an assumption belongs to some Σ_{m_j}. Since both formal proofs (of alpha and neg alpha) are finite. there is some i^* bigger than all of these m_i's and m_j's. Therefore for each β_i or γ_j that are used as assumptions $\beta_i, \gamma_j \in \Sigma_{i^*}$. Now $\Sigma_{i^*} \vdash \alpha$ and $\Sigma_{i^*} \vdash \neg\alpha$. So Σ_{i^*} is inconsistent, contradict 2 in our construction.

Theorem Every consistent Σ is satisfiable

proof Let Σ' be a max consistent set of wffs st $\Sigma \subseteq \Sigma'$. Define a truth assignment $v_{\Sigma'}$ as follows:

$$v_{\Sigma'} = \begin{cases} T & \text{if } \Sigma; \vdash p_i \\ F & \text{otherwise} \end{cases}$$

Claim For every formula α, $v_{\Sigma'}(\alpha) = T$ iff $\Sigma' \vdash \alpha$.

Chapter 5

L13

The completeness thm Every consistent Σ is satisfiable.

pf First step: pick $\Sigma' \supseteq \Sigma$ which is maximally consistent.
Second: Define a truth assignment $V_{\Sigma'}$ as follows, for any prop variable p,

$$V_{\Sigma'}(p) = \begin{cases} T & \text{if } \Sigma' \vdash p \\ F & \textit{otherwise} \end{cases}$$

Claim For every formula α, $\Sigma' \vdash \alpha$ iff $\overline{v}_{\Sigma'}(\alpha) = T$

Proof of the claim By generalized induction on the set of all wffs

$$I(Prop\ variables, \{\rightarrow, \neg\})$$

Base $\alpha = p$ for some prop. var. p,
If $\Sigma' \vdash p$, by def of $V_{\Sigma'}$, $V_{\Sigma'}(p) = T$.
If $\Sigma' \nvdash p$, then $V_{\Sigma'}(p) = F$.

Induction Step Assume the claim holds for α and for β, need to show it for $(\neg\alpha)$ and $(\alpha \rightarrow \beta)$.

- First case: $(\neg\alpha)$.
  if $\Sigma' \vdash \neg\alpha$ by the consistency of Σ', $\Sigma' \nvdash \alpha$. So by ind. hyp. $\overline{V}_{\Sigma'}(\alpha) = F$. So by truth table of $\neg$, $\overline{V}_{\Sigma'}(\neg\alpha) = T$.

  if $\Sigma' \nvdash (\neg\alpha)$, then by its maximallity, $\Sigma' \vdash \alpha$. so by ind. hyp, $\overline{V}_{\Sigma'}(\alpha) = T$, then $\overline{V}_{\Sigma'}(\neg\alpha) = F$.

- second case $(\alpha \rightarrow \beta)$. if $\Sigma' \vdash \alpha \rightarrow \beta$. Either $\Sigma' \vdash \alpha$ in which case $\Sigma' \vdash \beta$.
  Using the ind. hyp, we get $\overline{V}_{\Sigma'}(\alpha) = T$, and $\overline{V}_{\Sigma'}(\beta) = T$, so by truth table if $\rightarrow$, $\overline{V}_{\Sigma'}(\alpha \rightarrow \beta) = T$.

  otherwise $\Sigma' \nvdash \alpha$, so by the ind. hyp, $\overline{V}_{\Sigma'}(\alpha) = F$, then $\overline{V}_{\Sigma'}(\alpha \rightarrow \beta) = T$.

Assume $\Sigma' \nvdash \alpha \rightarrow \beta$.
Then $\Sigma' \nvdash \beta$, if it does, we can use the axiom $\beta \rightarrow (\alpha \rightarrow \beta)$ and MP to get $\Sigma' \vdash (\alpha \rightarrow \beta)$ contradiction. So by ind. hyp, $\overline{V}_{\Sigma'}(\beta) = F$.
We need to show $\overline{V}_{\Sigma'}(\alpha \rightarrow \beta) = F$. Since we know already $\overline{V}_{\Sigma'}(\beta) = F$. Our claim holds unless $\overline{V}_{\Sigma'}(\alpha) = F$.
Now by ind. hyp, this implies that $\Sigma' \nvdash \alpha$, so by maximality, $\Sigma' \vdash \neg\alpha$, then $\Sigma' \vdash (\alpha \rightarrow \beta)$. Contradiction.

Subclaim If $\Sigma' \vdash \alpha$ then $\Sigma' \vdash (\alpha \rightarrow \beta)$. Use deduction, suffices to show that $\Sigma \cup \{\alpha\} \vdash \beta$ iff $\Sigma \cup \{\alpha\} \cup \{\neg\alpha\} \vdash \beta$

Completeness thm for all α and any set of wffs Σ. If $\Sigma \models \alpha$ then $\Sigma \vdash \alpha$.

Proof Otherwise, then $\Sigma \cup \{\neg\alpha\}$ is consistent. So by out last result, $\Sigma \cup \{\neg\alpha\}$ is satisfiable, so $\Sigma \not\models \alpha$.

Claim If Σ is maximally consistent, then it is maximally satisfiable.

Proof consistent $\implies$ satisfiable
Since for every α, $\Sigma \vdash \alpha$ or $\Sigma \vdash \neg\alpha$. We get by soundness, $\Sigma \models \alpha$ or $\Sigma \models \neg\alpha$.
If Σ is maximally satisfiable, satisfiable $\implies$ consistent. Since for every α, $\Sigma \models \alpha$ or $\Sigma \vdash \neg\alpha$. We get by completeness, $\Sigma \vdash \alpha$ or $\Sigma \vdash \neg\alpha$, so Σ us maximally consistent.

Recall Compactness thm Σ satisfiable iff every finite $A \subseteq \Sigma$ is satisfiable.

One big question: is there a polynomial time algorithm to figure out if a given α is satisfiable, P vs NP

---

lengths of proofs: Is there a polynomial p(n) such that for every tautology α, there is a formal proof of $\leq p(|\alpha|)$ length?

Existence of proof system.

Chapter 6

L14

p: Every man is mortal.
q: Socrates is a man.

---

s: Socrates is mortal.

$\{p, q\} \vDash s$

6.1 First order logic (predicate calculus)

Objects	properties
man	mortal
socrates	

Actually we shall discuss a large family of languages.

Common to all logical symbols
$\rightarrow, \wedge, \vee, \neg$ propositional connectives
$\forall, \exists$ quantifiers
$=$ equality

Language specific symbols constant symbols
relation symbols
function symbols

6.1.1 (Informal) examples of specific languages

1. A language for number theory

constant symbols	0, 1	a, b
relation symbols	$\leq$	R
function symbols	+, *	f, g

   $\phi(x) \equiv \forall y \forall z\,(g(y,z) = x \rightarrow y = x \vee z = x)$ "x is a prime number"
   $\psi(x) \equiv \exists z f(z,z) = x$ "x is an even number"

   " there are ∞ many prime numbers" $\forall x \exists y\,(R(x,y) \wedge \phi(y))$

2. A language for set theory

relation symbols $\in$ R $R(x,y)$ stands for $x \in y$
$\phi(x) =$ "x is an empty set." $\forall y(\neg R(y,x))$
"there is only one empty set" $\forall x \forall y(\phi(x) \wedge \phi(y) \rightarrow x = y)$
$x \subseteq y$ $\forall z(R(z,x) \rightarrow R(z,y))$

What can we say in the minimal language. No constant symbols, no function symbols, no relation symbols.

$$\forall x \forall y (x = y) \equiv \text{“there is only one element in my universe”}$$

there are more than 2 elements $\exists x \exists y \exists z\, (\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z))$
there are at most 2 elements $\forall x \forall y \forall z\, (x = y \vee y = z \vee x = z)$

syntax	semantics
what is wff	$\alpha \vDash \beta$
proof system	Σ is satisfiable
consistency	
$\Sigma \vdash \alpha$	
$\vdots$	

Fix a language
constant symbols $a_1 a_2 \ldots$
function symbols $f_1, f_2, \ldots$
arity 1, 1, 3, 2
relation symbols $R_1, R_2 \ldots$
arity 1, 1, 2

- Step 1: Define the collection of "words" that denote objects - terms

  the set of Terms is defined as $I(A,P)$ where A- all constant symbols and all variable symbols $x_1 x_2 \ldots$
  P $\{O_f : f$ is a function symbol$\}$ $\frac{t_1,t_2}{f(t_1,t_2)}$ $\frac{t}{f(t)}$

  Examples of terms

  1. language for number theory
     $a, b, x_1 x_2 \ldots$ $f(a,b), g(x,a), f(x,y)$ $f(g(x,a),y)$
     $g(f(b,b),x) \mapsto 2x$
     $f(g(x,x), f(f(y,y),y)) \mapsto x^2 + 3y$
  2. The terms of the language of set theory
     No function symbols, therefore P is empty. $I(A,P) = A$
  3. The terms in the empty language
     Same- just variable symbols

- Step 1.5: defining the set of atomic formulas

  $$\{R(t_1, \ldots .t_k) : \text{R is a k-ary relation symbol, and } t_1 \ldots t_k \text{ are terms}\}$$

  Example

  1. number theory
     $R(a,b)$ $(0 \leq 1)$
     $R(f(a,x), g(f(b,b),y))$ $x + 0 \leq 2y$
  2. set theory
     $y \in X \cap Z$ $R(y,x) \wedge R(y,z)$ (not atomic)
     $y = z$ $x \in y$

- Step 2: Defining our wff's, again as $I(A, P)$
  The set of wffs in a given language L (given constants, function, relations)

$$I(\text{atomic formulas}, \{\wedge, \vee, \to, \neg, O_\forall, O_\exists\})$$

$$O_{\forall x}\ \frac{\phi}{\forall\phi} \qquad O_{\exists y}\ \frac{\phi}{\exists y\phi}$$

$L \implies wff's\ of\ L$

$2x + y^2$ term
~~$\forall x(2x + y^2)$~~
$x(2x + y^2 = x)$ atomic formula
$\forall x(2x + y^2 = x)$ formula (not atomic)

6.1.2 Important syntactic notion - free variable

We define by induction on the construction on the set of wff's, $F(\phi)$ - the set of free variables of ϕ. If ϕ is atomic - $F(\phi)$ = all variables occurring in ϕ.

Examples $F(\underbrace{f(b,b)}_{t_1} = \underbrace{g(x, f(b,y))}_{t_2})) = x, y$
$F(\phi_1 \wedge \phi_2) = F(\phi_1 \vee \phi_2) = F(\phi_1 \to \phi_2) = F(\phi_1) \cup F(\phi_2)$
$F(\phi) = F(\neg\phi)$
$F(\forall x\phi) = F(\phi) \setminus \{x\}$ similar for $\exists$

Chapter 7

L15

Recall logical symbols: brackets, connectives, quantifiers, equals, variables.
language-dependent: constants $(c_1, c_2, \ldots)$ functions, relations (R_1, R_2)

Defn (Terms)

$$I(\{c_1, c_2, \ldots, x_1, \ldots\}, \{f_1, f_2 \ldots\})$$

"atomic formulas" $\{R_i(t_1, \ldots, t_k) : R_i$ is a k-ary relation and t_j are terms $(1 \leq j \leq k)\}$
"wff" $I(\text{atomic formulas}, \{\neg, \wedge, \vee, \rightarrow\} \cup \{\forall, \exists\})$
"free variable" of ϕ, some variable not in the scope of $\forall, \exists$ somewhere in ϕ.

eg $\forall x(R(x, y))$ $\{y\}$ is free.
$((\forall x(R(x, y)) \wedge P(x)))$ $\{x, y\}$
$\forall x \exists y f(x, y, z)$ not a wff

Defn if ϕ has no free variables, ϕ is a sentence.

Defn Given some (syntactical) language $[c_1, \ldots \text{constants}, f_1, \ldots \text{functions}, R_1, \ldots \text{relations}]$, a "structure" consists of:

- a universe $\mathcal{U}$ (domain), non-empty!
- give an element in $\mathcal{U}$ to each constant c_i
- a mapping for each $f_i : \mathcal{U}^k \rightarrow \mathcal{U}$ (f_i k-ary)
- a relation for each $R_i : \mathcal{U}^k \rightarrow \{T, F\}$ (R_i k-ary)

number theory language
constants: a, b
functions: f, g
relation: R
All above are syntax

structure:
$\mathcal{U} = \mathbb{N}$ $\quad a = 0, b = 1 \quad f(x, y) = x + y, \quad g(x, y) = x \cdot y \quad R(x, y) = T$ iff $x \leqslant y$

defn An assignment function $s : V \rightarrow \mathcal{U}$

Extend s to $\overline{s}$: $T \to \mathcal{U}$

Extent $\overline{s}$ (one last time) to handle any wff ϕ.

1. $\phi = (\neg\eta), \overline{s}(\phi) = T \iff \overline{s}(\eta) = F$
2. $\phi = (\eta \to (\wedge\vee)\psi)$ then use truth table

wlog consider $\phi = \forall x_i(\eta)$, define $\overline{s}_i^d(x) = \begin{cases} \overline{s}(x) & \text{if } x_i \neq x \\ d & \text{if } x_i = x \end{cases} \qquad d \in \mathcal{U}$

$\overline{s}(\phi) = T \iff$ for all $d \in \mathcal{U}$ we have $\overline{s}_i^d(\eta) = T$

Thm Any wff has a unique decomposition

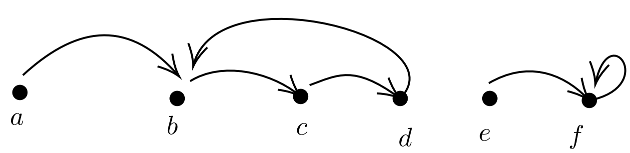
Cor $\overline{s}$ is well-defined.

7.1 graph theory

Define a language for graph theory.

syntax

- constants to be our vertices
- no functions
- R, P relations



structure

$\mathcal{U} = \{a, b, c, d, r, e, f\}$
$R(u, v) = T \iff (u, v)$ is a directed edge
$P(u, v) = T \iff$ u to v is connected

$R(u, v) \to P(u, v)$
$\forall u \forall v((P(u, v) \wedge \neg R(u, v)) \to \exists z(P(u, z) \wedge P(z, v) \wedge \neg(u = z) \wedge \neg(v = z)))$
These two fully defines P

Chapter 8

L16

8.1 Semantics for 1st order logic

Given a language $L =< R, F, a >$
Semantics for the first order logic over L is a rule that assigns T or F to every wff.

Need to fix

1. A structure for L: a universe set and interpretations for the symbols of L

   Example $L =< R, F(), G(), a, b >$
   $M_1 =< \mathbb{N}, \leq, +, \times, 0, 1 >$
   $\phi \equiv \forall x \exists y (\neg(x = y) \wedge R(y, x))$ false in M_1, $M_1 \nvDash \phi$

2. An assignment of variables to elements of the structure's universe

Lemma Let ϕ be a wff in some language L and M a structure for L, then for every assignments s_1, s_2 (to the universe of M) if for every variable that occurs free in ϕ, $s_1(x) = s_2(x)$. Then $M \vDash_{s_1} \phi$ iff $M \vDash_{s_2} \phi$

Cor If ϕ is a sentence (no free var) then for any s_1, s_2, $M \vDash_{s_1} \phi$ iff $M \vDash_{s_2} \phi$. Therefore, when we discuss truth values of sentences we do not specify any assignments.

Proof of the Lemma By generalized induction on the structure of ϕ.

Base step Atomic formulas: $(t_1 = t_2)$ or $R(x, y, a)$
(Note that for every term t, if s_1, s_2 agree on the variables in t, then $\overline{s}_1(t) = \overline{s}_2(t)$)

Induction step $\rightarrow, \wedge, \vee, \neg \forall, \exists$.

8.2 semantic notions

1. ϕ is a logical truth if for every structure for the language of ϕ and every assignment s to that structure $M \vDash_s \phi$
2. Σ logically implies ϕ if for every M, s, that make every member of Σ get true. $M \vDash_s \phi$

Lemma For every Σ, α, β, $\Sigma \cup \{\alpha\} \vDash \beta$ iff $\Sigma \vDash (\alpha \to \beta)$

Cor $\exists y \forall x R(x, y) \vDash \forall x \exists y R(x, y)$