AMATH 351

Sibelius Peng

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Contents

CHAPTER 1

Theory of Second-order Linear DEs

1. List of solution strategy for DEs
   (a) Method od undetermined coefficients
   (b) Integration Factor
   (c) Separable DEs
   (d) Variation of Parameter
   (e) Reduction of order
2. Approximation Method
   (a) Perturbation Method
   (b) Series Solution
   (c) Numerical Methods
      - Newton's Method
      - Euler's Method
      - Runge-Kutta Method

Comment Some reasons why we study linear second order ODEs

- have some various applications to nature (most, frequently used)
- the general theory extend quite naturally to higher order linear DEs
  i.e. we can write all linear DEs as a system of first order DEs

1.1 Classification of DEs

Defn A DE involves at least one independent variable (say x) and a dependent variable (say y) and their derivatives. If such DE only has one independent variable, it is called ordinary DE (ODE). A general form of n^{th} order ODE is of the form

$$F\left(\frac{d^n y}{dx^n}, \ldots, \frac{dy}{dx}, y, x\right) = 0$$

Defn We say that ODE is linear if

1. y or any of its derivatives appear only to the first power
2. y or any of its derivatives are not multiplied by any of y or its derivatives
3. y or any any of its derivatives are not arguments of any nonlinear functions.

Linear ODEs are of the form

$$a_n(x)\frac{d^n y}{dx^n} + \ldots + a_1\frac{dy}{dx} + a_0 y = f(x)$$

The above equation is homogeneous if $f(x) = 0$ with constant coefficient. All $a_i(x)$ constant

Examples

1. $y' = 5$ linear first order
2. $xy' = 5$ linear first order
3. $yy' = 5$ nonlinear first order
4. $y'' + x \ln y = 0$ nonlinear second order

1.2 Second order linear DEs

The most general form of SOLDE is

$$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

Assume that $a_2(x) \neq 0$. We divide by $a_2(x)$

$$\Longrightarrow \frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

where $P(x) = \frac{a_1(x)}{a_2(x)}, Q(x) = \frac{a_0(x)}{a_2(x)}, R(x) = \frac{f(x)}{a_2(x)}$

The associated homogeneous Equation is

$$y''(x) + P(x)y + Q(x)y = 0 \qquad (1)$$

Normal Form of Homo. DE: $u'' + qu = 0$

We define
$y(x) = u(x)v(x)$
$y' = u'v + v'u$
$y'' = u''v + v''u + 2u'v'$

Sub all above into (1)

$$(u''v + v''u + 2u'v') + P(x)(u'v + uv') + Q(x)uv = 0$$

$$vu'' + (vP(x) + 2v')u' + (v'' + P(x)v' + Q(x)v)u = C \qquad (2)$$

We pick $v(x)$ such that $Pv + 2v' = 0 \implies \frac{v'}{v} = -\frac{P(x)}{2}$

$$v(x) = e^c \exp\left(-\int \frac{P(x)dx}{2}\right)$$

Put $C = 0$,

$$v(x) = \exp\left(-\int \frac{P(x)dx}{2}\right)$$

eq (2) becomes

$$u'' + \left(\frac{v''}{v} + P\left(\frac{v'}{v}\right) + Q\right)u = 0$$

$2v' + Pv = 0$ diff wrt x,
$2\frac{v''}{v} + P' + P\frac{v'}{v} = 0$
$\frac{v''}{v} = \frac{P^2}{4} - \frac{P'}{2}$
$\vdots$

$$u'' + \left(\underbrace{Q(x) - \frac{P'}{2} - \frac{P^2}{4}}_{q(x)}\right)u = 0$$

$$\implies u'' + q(x)u = 0 \qquad \text{Normal form of Homo. eq}$$

where $q(x) = Q(x) - \frac{P'(x)}{2} - \left(\frac{P(x)}{2}\right)^2$

Theorem Existence and Uniqueness Thm for SOLDE
The SOLDE is in the form

$$y'' + P(x)y' + Q(x)y = R(x)$$

Let $P(x), Q(x), R(x)$ are continuous functions in closed interval $[a, b]$. If $x_0 \in [a, b]$, and if $y(x_0)$ and $y'(x_0)$ are any numbers, then the above DE has only one solution on the entire interval such that initial conditions are satisfied.

General Solution of SOLDE $y(x) = y_h + y_p$

Step 1 Find the general solution of homogeneous solution, $y_h = c_1 y_1 + c_2 y_2$, where $c_1, c_2 \in \mathbb{R}$, and y_1 y_2 are linearly independent solutions.

Step 2 particular solutions (1) variation of parameter (2) method of undetermined coefficient

Def Let y_1, y_2 be 2 solutions of SOLDE. We define Wronskian of 2 solutions as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

We have $W(y_1, y_2) \neq 0 \implies y_1 \& y_2$ are LI solutions
If $W(y_1, y_2) = 0$ then y_1, y_2 are linearly dependent solutions. i.e. $y_2 = \alpha y_1$

Theorem (Uniformity and Wronskian)
If $y_1(x)$ and $y_2(x)$ are solutions of homogeneous problem, $y''(x) + P(x)y' + Q(x)y(x) = 0$, then the Wronskian is either 0 or never zero on the given interval $[a, b]$

Proof We have

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \end{aligned}$$

Diff wrt x

$$W' = y_1 y_2'' - y_2 y_1''$$

Since $y_1 \& y_2$ are solutions of homogeneous problem,

$$y_1'' = -P y_1' - Q y_1$$

$$y_2'' = -P y_2' - Q y_2$$

Take them into W', we got $W' = -P[y_1 y_2' - y_1' y_2] = P(x)W$
$\implies W = W_0 \exp\big(-\int P(x)dx\big)$ where $W_0 = e^c$ for some arbitrary constant c.
Then W depends on W_0, so it is either zero or never zero on given interval $[a, b]$.

Lemma (Linear Dependence & Wronskian)
If $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous problem, then they are LD on given interval $[a, b]$ iff $W(y_1, y_2) = 0$.

Proof Suppose y_1, y_2 are LD. Then $y_2 = \alpha y_1$ where α is a constant. Then

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & \alpha y_1 \\ y_1' & \alpha y_1' \end{vmatrix} = 0$$

Now suppose that $W(y_1, y_2) = 0$, then $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$. Since the determinant is zero, then the matrix is singular, so one of the column in the scalar multiple of the other column. Then $y_2 = \alpha y_1$. So $y_1 \& y_2$ are LD.

Example Show that $y = c_1 \sin x \sin x + c_2 \cos x$ is GS of $y'' + y = 0$ on any interval. Find y_p for $y(0) = 2$ and $y''(0) = 3$.

Solution Let $y_1 = \sin x, y_2 = \cos x$. Can verify that $y_1'' + y_1 = 0, y_2'' + y_2 = 0$. From superposition principle, GS of given DE in

$$y = c_1 y_1 + c_2 y_2 = c_1 \sin x + c_2 \cos x$$

$$y' = c_1 \cos x - c_2 \sin x$$

Using ICs, then we can know $c_2 = 2, c_1 = 3$, then

$$y_p = 3 \sin x + 2 \cos x$$

1.3 (Initial Value Problem) IVPs VS (Boundary Value Problems) BVPs

ODEs can be classified into IVPs and BVPs. The equation themselves can be same, what differs are the conditions that are imposed to determine the unknown constants.

For IVP, 2 conditions are imposed at the same time. Time is independent parameter.

eg: $y(0) = \alpha, y'(0) = \beta$.

For BVP, 2 conditions are imposed at different time or locations. In general, we pick space coordinates independent parameter.

eg: $y(0) = \alpha, y'(1) = \beta$

1.4 Reduction of Order - SOLDE

The idea is to use the known solution to find another solution. The homogeneous problem is
$y'' + P(x)y + Q(x)y = 0$ (1)
Let $y_1(x)$ be one of the solution for eq(1). Let's assume the second solution is of the form

$$y_2(x) = v(x)y_1(x) \qquad (2)$$

Where $v(x)$ is an unknown function.

$$y_2' = v'y_1 + vy_1' \qquad (3)$$

$$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1'' \qquad (4)$$

Sub 2,3,4 into 1, we obtain,

$$\implies y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

collect the terms, then

$$v(y_1'' + P(x)y_1' + Q(x)y_1) + v'(2y_1' + Py_1) + v''(y_1) = 0$$

then

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - P(x)$$

then

$$\ln|v'| = -2\ln|y_1| - \int P(x)dx + C$$

$$v' = \frac{A}{y_1^2}\exp\Big(-\int P(x)dx\Big)$$

where $A = \pm e^C$

WLOG, take $C = 0, A = 1$, then ...

$$v = \int \frac{1}{y_1^2}\exp\Big(-\int_0^x P(s)ds\Big)dx \qquad (6)$$

so

$$y_2(x) = y_1\int \frac{1}{y_1^2}\exp\Big(\int P(s)ds\Big)dx$$

where $y_1(x)$ in the solution of Eq(1) (Homogeneous problem)

$$W(y_1, y_2) = \ldots = e^{-\int P(x)dx} \neq 0 \implies \textit{independent solutions}$$

Example Let $y_1(x) = \sin x$ be one of solutions of DE $y'' + y = 0$. Find another solution using reduction of order.

Soln Suppose $y_2(x) = v(x)y_1(x) = v(x)\sin(x)$ (1)
be the second solution of DE $y'' + y = 0$ (2)
$y_2' = v\cos x + v'\sin x$ (3)
$y_2'' = v'\cos x - v\sin x + v'\cos x + v''\sin x$
$y_2'' = v''\sin x + 2v'\cos x - v\sin x$ (4)
Sub 1,3,4 into 2, (after some cancellation)

$$\frac{v''}{v'} = -2\frac{\cos x}{\sin x}$$

then solve it, we have

$$v' = A\csc^2(x) \qquad A = \pm e^c$$

$v(x) = -A\cot x + B$ WLOG, $B = 0, A = -1$, then $v(x) = \cot x$, then $y_2(x) = v(x)y_1(x) = \cos x$

Example Let $y_1(x) = x^2$ be one of the solution the DE $x^2y'' + xy' - 4y = 0$. Find the second solution using reduction of order.

Soln We write this E in standard form. $x \neq 0$, divide by x.

$$y'' + \frac{1}{x}y' - \frac{y}{x^2}y = 0 \qquad (1)$$

Compose it with standard form,

$y'' + P(x)y' + Q(x)y = 0 \qquad (2)$
Use the formula, we get

$$y_2(x) = v(x)y_1(x) = y_1(x)\int \frac{1}{y_1^2(x)} \exp\Big(-\int P(s)ds\Big)dx$$

then ...

$$y_2 = \frac{1}{x^2}$$

GS: $y = c_1y_1 + c_2y_2 = c_1x^2 + \frac{c_2}{x^2}$

1.5 Variation of Parameter

Let $y_1(x) \& y_2(x)$ be linearly independent solutions of a homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0 \tag{1.1}$$

Consider

$$y'' + P(x)y' + Q(x)y = R(x) \tag{1.2}$$

We assume y_p is of the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{1.3}$$

$$y_p' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$$

Choose $v_1 \& v_2$ such that

$$v_1'y_1 + v_2'y_2 = 0 \tag{1.4}$$

$$y_p' = v_1y_1' + v_2y_2' \tag{1.5}$$

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$$

$$\implies \ldots \implies v_1[y_1'' + P(x)y_1' + Q(x)y_1] + v_2[y_2'' + P(x)y_2' + Q(x)y_2] + v_1'y_1' + v_2'y_2' = R(x)$$

$$\begin{matrix} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = R(x) \end{matrix} \implies \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

Inverse of 2×2 non-singular matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ R(x) \end{bmatrix} = \frac{R(x)}{W} \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

where $W = y_1 y_2' - y_2 y_1'$

$$\implies v_1(x) = \int \frac{-y_2(x)R(x)}{W(y_1, y_2)} dx \qquad v_2(x) = \int \frac{y_1(x)R(x)}{W(y_1, y_2)} dx$$

$$y_p = v_1 y_1 + v_2 y_2 = y_1(x) \left[\int \frac{-y_2(x)R(x)}{W(y_1, y_2)} dx \right] + y_2(x) \left[\int \frac{y_1(x)R(x)}{W(y_1, y_2)} dx \right]$$

Particular solution of inhomogeneous DE $y'' + P(x)y' + Q(x)y = R(x)$

Example Find a particular solution of $y'' + y = \csc(x)$ using variation of parameter.

Sol The GS of $y'' + y = 0$ is $y_h = c_1 y_1 + c_2 y_2 \implies y_1 = \sin x, y_2 = \cos x$

$$\implies W(y_1, y_2) = \ldots = -1 \neq 0$$

Hence, y_1, y_2 are independent, then using the formula

$$y_p = \sin x \left[\int \frac{-\cos x \csc x}{-1} dx \right] + \cos x \left[\int \frac{\sin x \csc x}{-1} dx \right] = \sin x \ln|\sin x| - x \cos x$$

is the particular solution of $y'' + y = \csc x$

Ex Verify it!

$$\frac{d}{dx}|x| = \frac{x}{|x|}, x \neq 0$$

$$\frac{d}{dx} \ln|x| = \frac{1}{x}, x \neq 0$$

(these two by chain rule)

Soln

$$y_p = \sin x \ln|\sin x| - x \cos x$$

$$y_p' = \cos x \ln|\sin x| + x \sin x$$

$$y_p'' = -\sin x \ln|\sin x| + \frac{\cos^2 x}{\sin x} + x \cos x + \sin x$$

$$y_p'' + y_p = \ldots = \frac{1}{\sin x} = \csc x$$

1.6 Example of 2^{nd} order ODEs with non-constant coefficients

1. Bessel's Equation
$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad p \in \mathbb{Z}$$
It determines the radial structure of the solution to Laplace's equation in both polar & spherical polar coordinates.

2. Legendre's Equation
$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0, \qquad p \in \mathbb{Z}$$
It determines the angular structure of the solution to Laplace's equation in both polar & spherical polar coordinates.

3. Laguerre's Equation
$$xy'' + (1 - x)y' + ay = 0, \qquad a \in \mathbb{R}$$
It represents the radial part of eigenfunction for hydrogen atom.

4. Hermite's Equation
$$y'' - 2xy' + 2ay = 0, \qquad a \in \mathbb{R}$$
it represents the set of eigenfunction for quantum mechanical harmonic oscillator.

CHAPTER 2

Series Solution and Special Functions

We will construct the power series to SOLDE with non-constant coefficients.

Def Transcendental Function:
Elementary functions that consists of algebraic functions such as trig, exponential, log and their inverses with operations: addition, subtraction, multiplication, and division are called Transcendental Function.

Ex

$$y = \tan\left[\frac{xe^{-x} + \ln(x^2+1)}{\arcsin(1+3x^2) - \ln(x^2+5) + |x+3|}\right]$$

Def Special function: Any function that is not a transcendental function is called special function.

Ex Bessel's function, Hermite Function

2.0.1 Review

Def A power series in x about x_0 is defined yo be

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + \dots$$

Normally, we set $x_0 = 0$, however, we can pick $x_0 \neq 0$.

Def A series is said to converge at x if

1. $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exists
2. The sum of the series is the value of the limit

Def Suppose a power series converges for $|x| < R$ for some $R > 0$, then R is called radius of convergence. Let us define its sum by $f(x)$ then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

Then the function $f(x)$ is smooth. In other words, it has continuous derivatives. Additionally, we can differentiate it term by term.

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + \dots$$

and

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2 a_3 x + \dots$$

Def A function $f(x)$ is said to be analytic at a point x_0 if there exists a power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$\forall x$ sufficiently close to x_0.

Equivalently, the function f is analytic at x_0 if its Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} (x - x_0)^n$$

converges to $f(x)$ $\forall x$ sufficiently close to x_0.

2.1 Series Solution to First order ODE

Consider $y' = y$ (1) [1]

We suggest a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \qquad (2)$$

[1] also can be solved by using separation of variable

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + \dots \qquad (3)$$

Sub 2 3 into 1

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n$$

Not in comparable form, then index shifting

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \xrightarrow{\text{index shifting}} \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n-1+1} = \sum_{n=0} a_{n+1} x^n$$

Then,

$$\sum_{n=0}^{\infty} \left((n+1)a_{n+1} - a_n\right) x^n = 0$$

This equation in the power series representation of zero. $\implies$ each of the coefficients in the series is exactly zero.

$$\implies a_{n+1} = \frac{a_n}{n+1}, \qquad n = 0, 1, 2, \dots$$

then ...

$$a_n = \frac{a_0}{n!} \implies y = a_0 \sum_{n=0} \frac{x^n}{n!}$$

If we have IC $y(0) = 1$

$$\implies y = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Defn Standard form of SOLDE (homogeneous) is

$$y'' + P(x)y' + Q(x)y = 0$$

The behavior of the solution near x_0 is completely determined by the behavior of $P(x)$ and $Q(x)$.

We say x_0 is an ordinary point if $P(x)$ and $Q(x)$ are analytic at x_0.

Defn Any point that is not an ordinary point is called singular point.

Ex

$$y'' + y = 0$$

Let's assume the power series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$, $|x| < R$ with $R > 0$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+1}x^n$$

$$y'' + y = 0 \implies \sum_{n=0}^{\infty}[(n+1)(n+1)a_{n+2} + a_n]x^n = 0 \implies (n+1)(n+1)a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)} \qquad n = 0, 1, 2, \dots$$

To specify a unique solution, we need to know $a_0 \& a_1$.

$$\dots \implies a_{2n} = (-1)^n \frac{a_0}{(2n)!}, \qquad a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$$

We can write

$$y = a_0 \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$y = a_0 \cos x + a_1 \sin x$$

Note If we assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

This is because $n = 0$ term of the derivatives is zero which means it does not change the value of the series. We can include the zero and not as needed.

$$\implies f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$f(x) = \sum_{n=0}^{\infty} a_n x^n$
$f''(x) = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$ by index shifting

Note

$$\left.\begin{array}{ll} y'' & \text{do index shifting} \\ x^2y'' & \text{use } \sum_{n=1}^{\infty}(n)(n-1)a_n x^{n-2} \end{array}\right\} \to \text{you are trying to get } \sum_{n=0}^{\infty}(\quad)x^n$$

y' – do index shifting

$2xy'$ – do not use index shifting

Ex find the series solution for the Legendre's equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \qquad \text{where } p \text{ is a constant}$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

Compare with $y'' + P(x)y + Q(x)y = 0$

$$P(x) = \frac{-2x}{1-x^2} \qquad Q(x) = \frac{p(p+1)}{1-x^2}$$

Here, $P(x)$ and $Q(x)$ are analytic about $x = 0$, $x = 0$ is an ordinary point.

$x = \pm 1$ is a singular point since $P(x)$ and $Q(x)$ are not defined (or analytic)

To find series solution we propose $y = \sum_{n=0}^{\infty} a_n x^n$. We need power series representation for $y'' - x^2 y' - 2xy'$ and $p(p+1)y$

$$\begin{aligned}
p(p+1)y &= \sum_{n=0}^{\infty} p(p+1)a_n x^n \\
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0} n a_n x^{n-1} \\
\implies -2xy' &= \sum_{n=0}^{\infty} (-2n a_n) x^n \\
y'' &= \sum_{n=2} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \\
\implies -x^2 y'' &= \sum_{n=0}^{\infty} -n(n-1) a_n x^n \\
y'' &= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \to \text{ index shifting}
\end{aligned}$$

Sub into the DE,

$$y'' - x^2 y'' - 2xy' + p(p+1)y = 0$$

$$= \sum_{n=0}^{\infty} ((n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n)x^n = 0$$

It is essential to write each term in the DE as a power series with the n^{th} term a multiple of x^n

$$\begin{aligned}
\implies (n+1)(n+2)a_{n+2} &= [n(n-1) + 2n - p(p+1)]a_n \\
a_{n+2} &= \frac{[n(n-1) + 2n - p(p+1)]a_n}{(n+1)(n+2)}
\end{aligned}$$

$$\begin{aligned}
-p(p+1) + n^2 + n &= -[p^2 + p - n^2 - n] \\
&= -[(p^2 - n^2) + (p - n)] \implies a_{n+2} = \frac{-[(p-n)(p-n+1)]}{(n+1)(n+2)} a_n \\
&= -[(p-n)(p+n+1)]
\end{aligned}$$

We need a_0 and a_1 to start with

- For even terms:

$$a_{2n} = \frac{(-1)^n \left[\prod_{i=0}^{n-1}(p-2i)\right] \left[\prod_{i=1}^{n}(p+2i-1)\right]}{(2n)!} a_0$$

- For odd terms:

$$a_{2n+1} = \frac{(-1)^n \left[\prod_{i=1}^{n}(p+2i)\right] \left[\prod_{i=0}^{n-1}(p-2i-1)\right]}{(2n+1)!} a_1$$

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \underbrace{a_{2n}}_{L_p^0} x^{2n} + \sum_{n=0}^{\infty} \underbrace{a_{2n+1}}_{L_p^1} x^{2n+1}$$

Lp^0: even Legendre's polynomial of order p
Lp^1: odd Legendre's polynomial of order p

$$L_p^0 = a_0 \left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} - \dots\right]$$
$$L_p^1 = a_1 \left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - \dots\right]$$

for

$$\begin{aligned} p &= 0 \qquad L_0^0 = 1 \\ p &= 1 \qquad L_1^1 = x \\ p &= 2 \qquad L_2^0 = 1 - 3x^2 \\ p &= 3 \qquad L_3^1 = x - \frac{5}{3}x^3 \end{aligned}$$

Theorem (Power Series Solution at ordinary part)
Let x_0 be the ordinary point of our standard homogeneous DE and let a_0 and a_1 are arbitrary constraints. Then there exists a unique function $f(x)$ that is analytic at x_0 and that is the solution of given DE in a certain neighbourhood of this point and it satisfies the ICs

$$y(x_0) = a_0, \qquad y'(x_0) = a_1$$

Note Let x_0 be a singular point.
Consider homogeneous DE $y'' + P(x)y' + Q(x)y = 0$ (1).
If either (or both) $P(x)$ & $Q(x)$ are not analytic at x_0.

Defn Suppose that x_0 is a singular point of Eq (1) such that $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at x_0, then x_0 is a Regular Singular Point (RSP). Otherwise, x_0 is an irregular singular point.

Ex Legendre's Eq

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0, \qquad p = ?$$

show that $x = \pm 1$ are regular singular point.

Soln Here, $P(x) = -\frac{2x}{1-x^2}, Q(x) = \frac{p(p+1)}{1-x^2}$
Consider $x = 1$, we notice that $P(x)$ and $Q(x)$ are not analytic at $x = 1$.

Consider $(x-1)P(x) = (x-1)\frac{2x}{x^2-1} = \frac{2x}{x+1}$

$$(x-1)^2 Q(x) = \frac{(x-1)p(p+1)}{(1+x)}$$

$\implies$ we see that $(x-1)P(x)\&(x-1)^2Q(x)$ are analytic at $x=1$.

$\therefore x = 1$ is RSP.

Example Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$
Check $x = 0$?

Soln In standard form,

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$$

$\implies P(x) = \frac{1}{x}, \qquad Q(x) = \frac{x^2-p^2}{x^2}$
At $x = 0$, $P(x)\&Q(x)$ are not analytic $\implies x = 0$ is a singular point.

$$xP(x) = 1, \quad x^2Q(x) = x^2 - p^2 \implies \textit{analytic}$$

So $x = 0$ RSP

To solve Bessel's equation about RSP $x = 0$, let us consider the associated Euler's equation

$$x^2y'' + p_0xy + q_0y = 0$$

where $p_0 = \lim_{x\to 0} xP(x) = 1$ and $q_0 = \lim_{x\to 0} x^2Q(x) = -p^2$

Note The general for associated Euler's equation is

$$ax^2 + bxy' + cy = 0$$

around $x = 0$

General Method for RSP

We get the general Method for Regular singular point $x = x_0$ as

$$(x-x_0)^2y'' + p_0(x-x_0)y' + q_0y$$

(associated Euler's Eq) where

$$\begin{cases} p_0 = \lim_{x\to x_0}(x-x_0)P(x) \\ q_0 = \lim_{x\to x_0}(x-x_0)Q(x) \end{cases}$$

We propose the general solution of above DE as $y = x^r$ into $[x^2y'' + p_0xy + q_0y = 0]$

$$\implies r(r-1)+p_0r+q_0=0 \qquad \text{Indicial eq}$$
$$\implies r^2+(p_0-1)r+q_0=0$$
$$\implies r_{1,2}=\frac{-(p_0-1)\pm\sqrt{(p_0-1)^2-4q_0}}{2}$$

We have 3 cases:

① Real & distinct roots ($r_1 \neq r_2$)
GS: $y(x)=c_1x^{r_1}+c_2x^{r_2}$

② Real & Equal ($r_1=r_2$)
GS: $y(x)=c_1x^{r_1}+c_2(\ln x)x^{r_1}$

③ Complex Roots $r_{1,2}=\alpha\pm i\beta$

$$\begin{aligned} x^r=e^{\ln x^r} &= e^{r\ln x} \\ &= e^{\ln x(\alpha\pm i\beta)} \\ &= e^{\alpha\ln x}e^{\pm i\beta\ln x} \\ &= e^{\alpha\ln x}[cos(\beta\ln x)\pm i\sin(\beta\ln x)] \end{aligned}$$

where $x>0$

Indicial Equation

$$r(r-1)+p_0r+q_0=0 \implies r_{1,2}=\frac{-(p_0-1)\pm\sqrt{(p_0-1)^2-4q_0}}{2}$$

Real and equal roots: $r_1=r_2$

$$(p_0-1)^2=4q_0$$

one of solution will be

$$y_1=x^{r_1}=x^{-\left(\frac{p_0-1}{2}\right)} \implies p_0=1-2r_1$$

Second solution using reduction of order

$$y_2(x)=v(x)y_1(x)=y_1(x)\left[\int\frac{1}{y^2}\exp\left(-\int P(x)dx\right)dx\right]$$

$$\implies v'=\ldots=\frac{1}{x} \implies v(x)=\ln|x|$$

For $x>0$, $v(x)=\ln x$, so $y_2(x)=vy_1=\ln(x)x^{r_1}$

GS

$$y=c_1x^{r_1}+c_2(\ln x)x^{r_1}$$

Method of Frobenius (for singular points)

In general, we define power series of the form

$$y(x)=x^r\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+r}$$

with $a_0 \neq 0$ and $a_n = 0 \quad \forall n < 0$.

Ex Consider $2x^2y'' + x(2x+1)y' - y = 0$ around $x = 0$.

Solution DE:

$$y'' + \left(\frac{\frac{1}{2}+x}{x}\right)y' - \left(\frac{1}{2x^2}\right)y = 0$$

on standard form

$$P(x) = \left(\frac{\frac{1}{2}+x}{x}\right) \qquad Q(x) = -\left(\frac{1}{2x^2}\right)$$

Now

$$xP(x) = \frac{1}{2} + x, \qquad x^2Q(x) = -\frac{1}{2} \implies \text{analytic at } x = 0$$

$$p_0 = \lim_{x\to 0} xP(x) = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2Q(x) = -\frac{1}{2}$$

Corresponding Indicial Equation is

$$r(r-1) + p_0 r + q_0 = 0 \implies r = 1, -\frac{1}{2}$$

We propose the series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0, a_n = 0 \text{ for } n < 0$$

We need expression for $2x^2y'', 2xy', xy', -y$

$$-y = \sum_{n=0}^{\infty} -a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$xy' = x\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r}$$

$$2x^2y' = 2x^2\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} = \sum_{n=1}^{\infty} 2(n+r-1)a_{n-1} x^{n+r}$$

$$2x^2y'' = 2x^2\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} = \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r}$$

Take them in to the DE
$\implies \ldots \implies 2(r-1)ra_0x^r + \sum_{n=1}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=1}^{\infty} 2(n+r-1)a_{n-1}x^{n+r} + (r-1)a_0x^r + \sum_{n=1}^{\infty}(n+r-1)a_n x^{n+r} = 0$

continued $r = 1, -\frac{1}{2}$

$$\implies \sum_{n=1}^{\infty} \left[(n+r-1)((2(n+r)+1)a_n + 2a_{n-1})\right] x^n = 0$$

$$n+r-1 \neq 0 \implies (2(n+r)+1)a_n + 2a_{n-1} = 0$$

Recursion Relation

$$a_n = \frac{-a_{n-1}}{n+r+\frac{1}{2}}, \qquad n = 1, 2, \dots$$

- For $r = 1$, $a_n = -\frac{a_{n-1}}{n+\frac{3}{2}}$

$$a_1 = -\frac{2}{5}a_0, \qquad a_2 = \frac{4}{35}a_0$$

- For $r = -\frac{1}{2}$, $a_n = -\frac{a_{n-1}}{n}$

$$a_1 = -a_0, \qquad a_2 = \frac{a_0}{2}$$

$\therefore$ 2 LI solution are

$$y_1 = x^1 \left(1 - \frac{2}{5}x + \frac{4}{35}x^2 - \dots\right)$$
$$y_2 = x^{-\frac{1}{2}} \left(1 - x + \frac{1}{2}x^2 - \dots\right)$$

Therefore, the GS

$$y = c_1 y_1 + c_2 y_2 = c_1 x \left(1 - \frac{2}{5}x + \frac{4}{35}x^2 - \dots\right) + c_2 x^{-\frac{1}{2}} \left(1 - x + \frac{1}{2}x^2 - \dots\right)$$

2.1.1 Extended Method of Frobenius

Around $x = x_0$ (R.S.P), we have associated Euler's Equation as $(x-x_0)^2 y'' + p_0(x-x_0)y' + q_0 y = 0$

$$p_0 = \lim_{x \to x_0} (x-x_0)P(x)$$
$$q_0 = \lim_{x \to x_0} (x-x_0)^2 Q(x)$$

By substitution,

$$y = (x-x_0)^r$$

We obtain Indicial eq:

$$r(r-1) + p_0 r + q_0 = 0$$

Consider $r_1 \geq r_2 \in \mathbb{R}$

1. There is one solution of the form

$$y_1 = (x-x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x-x_0)^n, a_0 \neq 0$$

2. If $r_1 - r_2 \neq \mathbb{Z}$, second LI solution will be

$$y_2 = (x - x_0)^{r_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n, b_0 \neq 0$$

3. If $r_1 = r_2 = r$, second LI solution using reduction of order is

$$y = y_1 \ln(x - x_0) + (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n, c_0 \neq 0$$

4. $r_1 - r_2$ is a positive integer, the second LI solution

$$y_2 = \alpha y_1 \ln(x - x_0) + (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad d_0 \neq 0$$

2.1.2 Bessel's Function

The solution of

$$x^2 y'' + xy' + (x^2 - p^2) y = 0$$

is called Bessel's function.

$x = 0$ RSP $\implies$ (standard form)

$$y'' + \frac{1}{x} y' + \frac{x^2 - p^2}{x^2} y = 0$$

$$p_0 = \lim_{x \to 0} x P(x) = \lim_{x \to 0} x \cdot \frac{1}{x} = 1$$

$$q_0 = \lim_{x \to 0} x^2 Q(x) = \lim_{x \to 0} x^2 \cdot \frac{x^2 - p^2}{x^2} = -p^2$$

Indicial Eq:

$$r(r-1) + p_0 r + q_0 = 0$$

$$r^2 - p^2 = 0 \implies r = \pm p$$

We consider $r = p > 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

We need expression for

$$x^2 y'', xy', x^2 y, -p^2 y$$

$$-p^2 y = \sum_{n=0}^{\infty} -p^2 a_n x^{n+r}$$
$$x^2 y' = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$
$$xy' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r}$$
$$x^2 y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r}$$

$$x^2y'' + xy' + x^2y - p^2y = 0 \implies \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \ldots = 0$$
$$\implies \sum_{n=2}^{\infty} a_{n-2}x^{n+r} + \sum_{n=0}^{\infty} a_n[(n+r)^2 - p^2]x^{n+r} = 0$$

since $a_0 \neq 0$ and $r = p > 0$

$$\implies \sum_{n=2}[a_{n-2} + a_n((n+r)^2 - p^2)]x^{n+r} + a_0(r^2 - p^2)x^r + a_1((r+1)^2 - r^2)x^{r+1} = 0$$

Since $r = p > 0$, and $2r + 1 \neq 0 \implies a_1 = 0$

We must have

$$a_{n-2} + a_n((n+r)^2 - r^2) = 0$$
$$-a_{n-2} = a_n(n(n+2r))$$
$$a_n = \frac{-a_{n-2}}{n(n+2r)}, \qquad n = 2, 3, 4, \ldots$$

We derived $a_1 = 0 \implies a_3 = a_5 = a_7 = 0$. So all odd term coefficients

$$\implies a_{2k+1} = 0, \qquad k = 0, 1, 2 \ldots$$
$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(1+r)\ldots(k+r)}$$

one of our solution is

$$y = a_0 x^p \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}k!(1+p)\ldots(k+p)}\right]$$

Bessel's Function To find second solution, we discuss different cases

1. $r_1 - r_2 = 2p$ is not an integer, second LI solution. Choose $p \to -p$

$$y_2 = a_0 x^{-p} \left[\sum_{k=0} \frac{(-1)^k x^{2k}}{2^{2k}k!(1-p)\ldots(k-p)}\right]$$

2. $r_1 = r_2$, use reduction of order.
3. $r_1 - r_2 = 2p$ is an positive integer, we can use reduction of order to find second LI solution.

2.2 Point at Infinity

$$y'' + P(x)y' + Q(x)y = 0$$

It is often desirable in physics, applied and pure math to study long time behaviour of the

$$y'' + P(x) + Q(x) = 0 \tag{2.1}$$

for a very large value of independent variable. For instance, if independent variable is time, we may want to know the solution of the system once the transient disturbances are faded away. In other words, $t \to \infty$.

We sub $t = \frac{1}{x}$ in the original equation and transform in given DE into a new variable.

$$t = \frac{1}{x} \implies \frac{dt}{dx} = \frac{-1}{x^2} = -t^2$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -t^2\frac{dy}{dt}$$

$$y'' = \ldots = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$$

Sub into original DE, we have

$$\vdots$$

$$\frac{d^2y}{dt^2} + \left[\frac{2}{t} - \frac{P\left(\frac{1}{t}\right)}{t^2}\right]\frac{dy}{dt} + \frac{Q\left(\frac{1}{t}\right)}{t^4}y = 0$$

$$\implies \ddot{y} + \left(\frac{2}{t} - \frac{P(1/t)}{t^2}\right)\dot{y} + \frac{Q(1/t)}{t^4}y = 0 \tag{2.2}$$

Here dot represent differentiation wrt t.

If we say Eq(2.1) has $x = \infty$ are ordinary point, a regular SP with $y = e^{rx}$ with exponential roots r_1 and r_2, or an irregular SP, then $t = 0$ is ordinary point, a regular SP, or irregular SP with Eq(2.2).

Example Consider

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = 0 \qquad \text{at } x = \infty$$

$$\implies x^2y'' + 4xy' + 2y = 0$$

what's the nature of the point?

sub $t = \frac{1}{x}$

$$\implies y' = -t^2y, \quad y'' = 2t^3\dot{y} + t^4\ddot{y}$$

Original DE becomes

$$t^4\ddot{y} + 2t^3\dot{y} + 4t(-t^2\dot{y}) + 2t^2y = 0$$

$$\ddot{y} - \frac{2}{t}\dot{y} + \frac{2}{t^2}y = 0 \quad \textit{near } t = 0$$

Here $P(t) = \frac{-2}{t}$ and $Q(x) = \frac{2}{t^2}$ are not analytic at $t = 0$, and

$$\lim_{t\to 0} tP(t) = -2, \quad \lim_{t\to 0} t^2 Q(x) = 2, \qquad \textit{analytic}$$

this implies $t = 0$ is a regular SP of equation 2.
$\implies x = \infty$ is a regular SP of equation 1

CHAPTER 3

Systems of First-order DEs - FODEs

Defn If $x_1(t), x_2(t), x_3(t), \ldots$ are unknown functions of a single variable t then the most general FODE in these unknown functions is,

$$\dot{x_1} = f_1(x_1, \ldots, x_n, t)$$
$$\dot{x_2} = f_2(x_1, \ldots, x_n, t)$$
$$\vdots$$
$$\dot{x_n} = f_n(x_1, \ldots, x_n, t)$$

In compact form,

$$\dot{\vec{x}}(t) = \vec{f}(\vec{x}, t)$$

or

$$\dot{\vec{y}} = \vec{f}(\vec{y}, t)$$

$n = 2$, the non-linear system:
example (Predator-Prey Model)
Lotta-Volterra Equation

$$\dot{x_1} = -ax_1 + bx_1x_2$$
$$\dot{x_2} = cx_2 - dx_1x_2$$

where $a, b, c, d > 0$, where

$$x_1 \to concentration\ of\ predator$$
$$x_2 \to concentration\ of\ prey$$

Example $n = 2$, linear system: Simple Harmonic Oscillator,

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -\omega^2 x_1$$

$$\ddot{x_1} = \dot{x_2} = \omega^2 x_1 \implies \ddot{x_1} + \omega^2 x_1 = 0$$

G.S: $x_1(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t)$

The most general form for a linear system of FODEs is

$$\dot{\vec{x}} = A(t)\vec{x} + \vec{b}(t)$$

or

$$\frac{d\vec{y}}{dt} = \dots$$

where $A(t)$ is an $n \times n$ matrix of coefficients $a_{ij}(t)$. $\vec{b}(t)$ is $1 \times n$ matrix of column vectors, with coefficients $b_i(t)$.

Reconsider the SHM example

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega^2 x_1 \end{aligned}\right\} \implies \begin{cases} A(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \\ \vec{b}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Any n^{th} order DE in function $f(t)$, linear or nonlineaer, can be expressed as system of FODEs as follows:

① We assume that DE in $y(t)$ can be

$$y^{(n)} = g(y, \dot{y}, \ddot{y}, \dots, y^{(n-1)}, t)$$

② Define

$$\begin{aligned} x_1(t) = y(t) &\implies \dot{x}_1(t) = \dot{y}(t) = x_2(t) \\ x_2(t) = \dot{y}(t) &\implies \dot{x}_2(t) = \ddot{y}(t) = x_3(t) \\ &\vdots \\ x_n(t) = y^{(n-1)}(t) &\implies \dot{x}_n(t) = y^{(n)}(t) = g(x_1, x_2, \dots, x_n, t) \end{aligned}$$

Why do we use system of FODEs?

① For theoretical reason: Existence-Uniqueness Theorem (Easy to prove).

② For practical purpose: (Easy to apply numerical methods).

System of FODEs

$$\frac{d\vec{y}}{dx} = \vec{f}(x, \vec{y}) \qquad \text{with ICs } \vec{y}(x_0) = \vec{y}_0$$

In 1D: $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$. If we consider $\frac{dy}{dx} = y, y(x_0) = y_0$.

G.S. $y(x) = y_0 e^x$

Consider $\frac{dy}{dx} = y^{\frac{1}{2}}, y(x_0) = y_0$
Unique Solution exists only if $y_0 > 0$.
$2y^{\frac{1}{2}} = x + c$. Solving if $c = 0$.
$y = \frac{x^2}{4}$ (two solution).
$y_0 = 0$ unique solution doesn't exist.

In short, we can relax the condition of differentiablility. We need a weaker form of ODE.

Weaker form of ODE

System of FODEs: $\frac{d\vec{y}}{dx} = \vec{f}(x, \vec{y}), \vec{y}(x_0) = \vec{y}_0$

In 1D, $\frac{dy}{dx} = f(x, y)$ with IC $y(x_0) = y_0$.
Integrating,

$$\int_{x_0}^{x} \frac{dy}{ds} ds = \int_{x_0}^{x} f(s, y(s)) ds$$

$$\implies y(x) - y(x_0) = \int_{x_0}^{x} f(s, y(s)) ds$$

or

$$y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s)) ds$$

weaker $\rightarrow$ no need to differentiate the function.

3.1 Picard's Method

We will construct an approximate solution to ODE while using the interval/weaker form of ODE.

To begin with

$$y_0(x) = y_0 \qquad (IC\ condition)$$

Roughest Approximation to the solution. It obeys only ICs. We will construct a solution using successive iterations.

$$y_1(x) = y_0 + \int_{x_0}^{x} f\left(s, y_0(s)\right) ds$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f\left(s, y_1(s)\right) ds$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f\left(s, y_{n-1}(s)\right) ds$$

Example $y' = x, y(0) = 1$. exact soln: $y = e^x$.
Picard's Approximation:

$$\begin{aligned}
y_0 &= 1 \\
y_1(x) &= 1 + \int_0^x y_0(s)ds = 1 + x \\
y_2(x) &= 1 + \int_0^x y_1(s)ds = 1 + x + \frac{x^2}{2} \\
y_3(x) &= 1 + \int_0^x y_2(s)ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \\
&\dots \\
y_n(x) &= 1 + \int_0^x y_{n-1}(s)ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}
\end{aligned}$$

$$\lim_{n\to\infty} = e^x$$

Example consider $y' = x + y, \quad y(0) = 1$.

$$\implies y(x) = -(x+1) + Ce^x, \quad y(0) = 1$$

$$\implies y(x) = -1 - x + 2e^x \qquad \text{exact solution}$$

by Picard's Approximation: we have

$$\begin{aligned}
y_0(x) &= 1 \\
y_1(x) &= 1 + \int_0^x [s+1]ds = 1 + x + \frac{x^2}{2!} \\
y_2(x) &= 1 + \int_0^x \left[1 + 2s + \frac{s^2}{2!}\right] ds = 1 + x + x^2 + \frac{x^3}{3!} \\
y_3(x) &= 1 + \int_0^x \left[1 + 2s + s^2 + \frac{s^3}{3!}\right] ds = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!} \\
y_4(x) &= 1 + \int_0^x \left[1 + 2s + s^2 + \frac{s^3}{3!} + \frac{s^4}{4!}\right] ds = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3 \cdot 4} + \frac{x^5}{5!} \\
&\vdots \\
y_n(x) &= 1 + x + 2\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}\right) + \frac{x^{n+1}}{(n+1)!}
\end{aligned}$$

$$\lim_{n\to\infty} y_n = -1 - x + 2e^x$$

3.2 Linear System and the Fundamental Matrix

3.2.1 Transforming a Scalar Equation to a system

We showed that any n^{th} order scalar equation (ODE) can be written as a system of n first order DEs. We show how to works for particular linear scalar equation of order n as follows. We have

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = g(x)$$

We define

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_n &= y^{(n-1)} \end{aligned}$$

we have

$$\frac{d^n y}{dx^n} = -P_n y_1 - P_{n-1}y_2 - \cdots - P_1 y_n + g(x)$$

$$\frac{d}{dx}\begin{bmatrix} y_1(x) \\ y_2(x) \\ \cdots \\ y_{n-1}(x) \\ y_n(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -P_n(x) & -P_{n-1}(x) & -P_{n-2} & \cdots & -P_1(x) \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ \cdots \\ y_{n-1}(x) \\ y_n(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ g(x) \end{bmatrix}$$

In general form

$$\frac{d\vec{y}}{dx} = A(x)\vec{y} + \vec{b}, \quad \vec{y}(x_0) = \vec{y}_0$$

Ex we consider general form of homogeneous second order DE

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' = -P(x)y' - Q(x)y$$

We define

$$\left.\begin{aligned} y_1 &= y(x) \\ y_2 &= y'(x) \end{aligned}\right\} \implies \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -Q & -P \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Theorem (Picard's Theorem for Linear System)
Let $A(x)$ and $b(x)$ be continuous functions on the closed interval $I \in [\alpha, \beta]$. Then there exists a unique solution to the IVP. So

$$\frac{d\vec{y}}{dx} = A(x)\vec{y} + \vec{b}(x), \qquad \vec{y}_0(x_0) = \vec{y}_0$$

where x_0 is the initial point in i and $\vec{y}_0$ is a constant vector with n-components. Solution exists throughout the interval I.

3.3 Homogeneous Linear System of Equations

Defn If A is a constant, it is called an autonomous system. If RHS of

$$\frac{d\vec{y}}{dx} = A(x)\vec{y}$$

depends on x explicitly, we say the system is non-autonomous.

Remark We define the solution $\vec{y}$ as a vector of the form $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$ where $\vec{y}$ is C^1 (continuously differentiable) vector-valued function is the given interval $I \in \mathbb{R}^n$.

Defn If A is constant, we define L (a linear operator) as $L\vec{y} = \left(\frac{d}{dx} - A\right)\vec{y}$. We define the space of solutions for $L\vec{y} = \vec{0}$ as the kernel of the linear operator L.

3.3.1 Solution space for system of DEs

From the linear algebra, we have

① Identity element

② Algebraic closure → Principle of Superposition

Theorem Let $\vec{y}' = A(x)\vec{y}$ where $A(x)$ is continuous on interval I. If a solution $\vec{y}$ satisfies $\vec{y}(x_0) = 0$ for some $x_0 \in I$, then $\vec{y}(x) = \vec{0}$ for all $x \in I$.

Theorem Let $\vec{y}' = A(x)\vec{y}$ where $A(x)$ is continuous on interval I. If solutions $\vec{y}_1(x), \ldots, \vec{y}_n(x)$ are linear independent at point $x_0 \in I$, then they are linear independent for all $x \in I$.

Def We define Wronskian for a system of equations as

$$W[\vec{y}_1(x), \ldots, \vec{y}_n(x)] = \det[\vec{y}_1(x), \ldots, \vec{y}_n(x)]$$

Theorem (Dimension of a solution space)
The solution space of $\frac{d\vec{y}}{dx} = A(x)\vec{y}$, where $A(x)$ is continuous in a n-dimensional vector space.

Remark A basis of the solution space is a set of n-solutions to DE which are linearly independent on given interval I.

That is, Wronskian of solution is nonzero on I. To construct a standard basis for solution space, we begin with standard basis of $\mathbb{R}^n$. i.e. $\{\vec{e}_1, \ldots, \vec{e}_n\}$ where $\vec{e}_j = \{0, \ldots, 0, 1, 0, \ldots, 0\}$ with only 1 in the j-th entry.

3.3.2 Basis for solution space

In $\mathbb{R}^n$, we write $\{\vec{e}_1, \ldots, \vec{e}_n\}$ as standard basis of $\mathbb{R}^n$. We take the standard basis vectors to be initial conditions for our DE and thus the standard basis for system of DE consists of

$$\{\vec{y}_1(x), \ldots, \vec{y}_n(x)\}$$

where the ICs at x_0 are $\vec{y}_j(x_0) = \vec{e}_j$ for $j = 1, 2, \ldots, n$.

Def The fundamental matrix at x_0 denoted by $\Phi(x, x_0)$ is

$$\Phi(x, x_0) = [\vec{y}_1(x), \ldots, \vec{y}_n(x)]$$

The fundamental matrix is formed with standard basis vectors as the columns of the matrix.

3.4 Finding solutions using eigenvalues

3.4.1 Review

We have the solution for homogeneous scalar equation as $y = e^{\lambda x}$. This may have real & distinct roots or real & repeated roots or complex conjugated roots.

To solve system of equations

$$\frac{d\vec{y}}{dx} = \vec{y}' = A\vec{y} \tag{3.1}$$

we need the solution of the form

$$\vec{y} = \vec{v}e^{\lambda x} \tag{3.2}$$

Sub (3.2) into (3.1)

$$\lambda\vec{v} = A\vec{v} \qquad e^{\lambda x > 0}$$

we find that $|A - \lambda I| = 0$ for non-trivial solution.

To find eigenvalue and eigenvectors. Then we write G.S. using superposition principle.

Ex Find the eigenvalue of matrix

$$A = \begin{bmatrix} 1 & 1 \\ y & 1 \end{bmatrix} \quad for \quad \frac{d\vec{y}}{dx} = A\vec{y}$$

Soln We consider the trial function $\vec{y} = e^{\lambda x}$. Sub it into given DE, we obtain

$$(A - \lambda I)\vec{v} = \vec{0}$$

Then λ must satisfy CE (characteristic equation), ... then $\lambda = -1, 3$.

For $\lambda = -1$, $\vec{v}_{\lambda=-1} = \binom{1}{-2}$.

For $\lambda = 3$, $\vec{v}_{\lambda=3} = \binom{1}{2}$

$$\vec{y} = c_1 e^{-x}\begin{pmatrix}1\\-2\end{pmatrix} + c_2 e^{3x}\begin{pmatrix}1\\2\end{pmatrix}$$

This solution can be determined completely in terms of finding c_1 & c_2 with given ICs.

To compute fundamental matrix, we must find 2 solutions that have ICs to be standard basis of $\mathbb{R}^2$. We pick

$$\vec{y}_1(0) = \vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\vec{y}_2(0) = \vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

Now

$$\vec{y}_1(0) = \begin{bmatrix}1 & 1\\-2 & 2\end{bmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \implies c_1 = c_2 = \frac{1}{2}$$

So

$$\vec{y}_1(x) = \begin{pmatrix}\frac{1}{2}e^{-x} + \frac{1}{2}e^{3x}\\ -e^{-x} + e^{3x}\end{pmatrix}$$

Similarly,

$$\vec{y}_2(0) = \begin{bmatrix}1 & 1\\-2 & 2\end{bmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = \vec{e}_2 \implies c_1 = -\frac{1}{4}, c_2 = \frac{1}{4}$$

$\therefore$

$$\vec{y}_2(x) = \begin{pmatrix}\frac{-1}{4}e^{-x} + \frac{1}{4}e^{3x}\\ \frac{1}{2}e^{-x} + \frac{1}{2}e^{3x}\end{pmatrix}$$

We obtain fundamental matrix as

$$\Phi(x, o) = [\vec{y}_1(x), \vec{y}_2(x)] = \ldots$$

Se sub $x = 0$, $\Phi(0, 0) = I_{2\times 2}$ as required.

Theorem Suppose the matrix A has n-eigenpairs, such that eigenvalues are real and distinct, then

$$\{e^{\lambda_1 x}\vec{v}_1, \ldots, e^{\lambda_n x}\vec{v}_n\}$$

forms the basis of solution space to the system of equations $\frac{d\vec{y}}{dx} = A\vec{y}$. Hence the G.S. is

$$\vec{y} = c_1 e^{\lambda_1 x}\vec{v}_1 + c_2 e^{\lambda_2 x}\vec{v}_2 + \ldots + c_n e^{\lambda_n x}\vec{v}_n$$

3.4.2 Finding Solution Using Eigenvalues for Complex Eigenvectors

We have the eigenvalues in complex conjugate pair. For the eigenvalue $\lambda = \mu + i\nu$, we write corresponding eigenvector (function) as $\vec{v}$ where $\vec{v} = \vec{a} + i\vec{b}$.
Complex eigenvector

$$\vec{u}(x) = e^{\lambda x}\vec{v},\ \vec{v}^* = \vec{a} - i\vec{b}$$

$$\vec{u}^*(x) = e^{\lambda^* x \vec{v}^*},\ \lambda^* = \mu - i\nu$$

We have

$$\vec{y}_1(x) = \text{Re}\{\vec{u}(x)\}$$
$$\vec{y}_2(x) = \text{Im}\{\vec{u}(x)\}$$

$$\vec{u}(x) = e^{\lambda x}\vec{v} + e^{(\mu+i\mu)x}(\vec{a}+i\vec{b}) = ... = e^{\mu x}(\vec{a}\cos(\nu x) - \vec{b}\sin(\nu x)) + ie^{\nu x}(\vec{a}\sin(\nu x) + \vec{b}\cos(\nu x))$$

So

$$\vec{y}_1(x) = \text{Re}\{\vec{u}(x)\} = e^{\mu x}(\vec{a}\cos(\nu x) - \vec{b}\sin(\nu x))$$
$$\vec{y}_2(x) = \text{Im}\{\vec{u}(x)\} = e^{\nu x}(\vec{a}\sin(\nu x) + \vec{b}\cos(\nu x))$$

where $\lambda = \mu + i\nu$, $\quad \vec{v} = \vec{a} + i\vec{b}$.

If all eigenvalues are real and distinct except 2 in complex conjugate pair, then G.S. is

$$\vec{y}(x)\,\underbrace{c_1\vec{y}_1(x) + c_2\vec{y}_2(x)}_{\text{complex conjugate}} + \underbrace{c_3e^{\lambda_3 x}\vec{v}_3 + \ldots + c_ne^{\lambda_n x}\vec{v}_n}_{\text{real and distinct}}$$

Ex Find the G.S. of vector DE

$$\frac{d}{dt}\vec{x} = \vec{x}' = A\vec{x}, \qquad A = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix}$$

Soln ... $\lambda = -1 \pm i$

When $\lambda = -1 + i$, $v = \binom{5}{-2+i}$.

We have the complex solution as $\vec{x} = ...$

$$\vec{x} = e^{-t}(\cos t + i\sin t)\left[\begin{pmatrix} 5 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = e^{-t}\left[\begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} + i\begin{pmatrix} 5\sin t \\ -2\sin t + \cos t \end{pmatrix}\right]$$

There are 2 solutions are linearly independent G.S. is

$$\vec{x} = e^{-t}\left[c_1\begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} + c_2\begin{pmatrix} 5\sin t \\ -2\sin t + \cos t \end{pmatrix}\right]$$

where c_1 and c_2 are arbitrary constants.

Ex Find the G.S. to the system

$$\frac{d}{dx}\vec{y} = \begin{bmatrix} -4 & 5 & -3 \\ \frac{-17}{3} & \frac{4}{4} & \frac{7}{3} \\ \frac{4}{3} & \frac{-25}{3} & \frac{-4}{3} \end{bmatrix}\vec{y}$$

Soln We need to find the solution of CE, $\det(A - \lambda I) = 0$

$$\implies \lambda = -2, -1 \pm 8i \qquad \text{Trial and Error method}$$

We have eigenvectors as

$$\vec{v}_1 = \begin{bmatrix} -1/2 + i/2 \\ -1/2 - i/2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1/2 - i/2 \\ -1/2 + i/2 \\ 1 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

From this we obtain the real & imaginary eigenvectors as

$$\vec{a} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

We get the G.S as

$$\vec{y} = c_1 e^{-x}(\vec{a}\cos(8x) - \vec{b}\sin(8x)) + c_3 e^{-2x}\vec{v}_3 + c_2 e^{-x}(\vec{a}\sin(8x) + \vec{b}\cos(8x))$$

3.4.3 Fundamental Matrix (Properties)

In case of autonomous system (A =constant) we have

1. Identity property:

$$\Phi(x_0, x_0) = [\vec{y}_1(x_0), \ldots, \vec{y}_n(x_0)] = [\vec{e}_1, \ldots, \vec{e}_n] = I_{n\times n}$$

2. $$\vec{y}(x) = \Phi(x)\vec{a}$$

   is a unique solution to IVP $\vec{y}' = A\vec{y}$ for $\vec{y}(0) = \vec{a}$

3. $\Phi(x)$ satisfies the matrix DE $\Phi'(x) = A\Phi(x)$

4. Multiplication Property

$$\Phi(x_1 + x_2) = \Phi(x_1) \cdot \Phi(x_2)$$

5. Inverse property

$$\Phi(-x) = [\Phi(x)]^{-1}$$

6. Time-invariant Property: If A = constant in linear system $\vec{x}' = A\vec{x}$, then if $\vec{x}(t)$ in the solution to the above DE, then $\vec{x}(t-a)$ is also a solution, $a \in \mathbb{R}$. We can also mention $\Phi(t, t_0) = \Phi(t - t_0, 0)$.

   In other words, it is only time-interval matters, not the starting or end points.

3.5 Exponential Matrix

Consider an autonomous system ($\vec{y}' = A\vec{y}$) with ICs $\vec{y}(0) = \vec{a} = \vec{y}_0$.

In scalar form, we have $y' = ay, y(0) = a = y_0$, the solution to IVP,

$$y(x) = y_0 e^{ax}$$

The solution to vector problem is

$$\vec{y} = \Phi(x, 0)\vec{a} = e^{xA}\vec{a}$$

We denote e^{xA} an exponential matrix where $A_{n\times n}$ is constant. Using Taylor's Series, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

We get

$$e^A = I + A + \frac{1}{2!}A^2 + \dots + \frac{A^n}{n!} + \dots$$

where both A and e^A are $n \times n$

3.5.1 Properties of Exponential Matrix

We consider $A_{n\times n}$ for $n = 2$. These properties can be generalized for $n \geq 2$.

1. $e^0 = I_{2\times 2}$
2. $e^{A+B} = e^A e^B$ is true only iff A and B commute, i.e. $AB = BA$
3. $e^{-A} = [e^A]^{-1}$ (inverse property)

---

1. Diagonal Matrix (2 real distinct roots)

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \implies e^A = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \frac{a^2}{2!} & 0 \\ 0 & \frac{b^2}{2!} \end{bmatrix} + \dots \\ &= \begin{bmatrix} 1 + a + \frac{a^2}{2!} + \dots & 0 \\ 0 & 1 + b + \frac{b^2}{2!} + \dots \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix} \end{aligned}$$

2. Upper triangular Matrix or (non-diagonalizable matrix)
(2 real and equal roots)

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \implies e^A = \begin{bmatrix} e^a & e^a \\ 0 & e^a \end{bmatrix}$$

we write

$$A = aI + N, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

where N is nilpotent matrix.

$$e^N = I + N + \frac{N^2}{2!} + \ldots = I + N$$

$$\begin{aligned} e^A &= e^{aI+N} = e^{aI}e^N = e^{aI}(I+N) \\ &= e^aI(I+N) = e^a(I+N) \\ &= e^a\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) \\ &= e^a\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\end{aligned}$$

3. Antisysmmetric matrix (complex conjugate pair)
   If $A = \begin{bmatrix}a & b\\-b & a\end{bmatrix}$, then $e^A = \begin{bmatrix}\cos b & \sin b\\-\sin b & \cos b\end{bmatrix} e^a$.

   We have $A = aI + B, B = \begin{bmatrix}0 & b\\-b & 0\end{bmatrix}$

   Since B and I commute,
   $$e^A = e^{aI+B} = e^{aI}e^B = e^ae^B$$
   where $e^B = I + B + \frac{B^2}{2!} + \ldots$
   $$\begin{aligned} B^2 &= \ldots = -b^2I \\ B^3 &= \ldots = \begin{bmatrix}0 & -b^3\\b^3 & 0\end{bmatrix} \\ B^4 &= b^4I\end{aligned}$$

   Hence
   $$e^B = \begin{bmatrix}1 - \frac{b^2}{2!} + \ldots & b - \frac{b^3}{3!} + \ldots\\-b + \frac{b^3}{3!} - \ldots & 1 - \frac{b^2}{2!} + \ldots\end{bmatrix} = \begin{bmatrix}\cos b & \sin b\\-\sin b & \cos b\end{bmatrix}$$
   $$e^A = e^a\begin{bmatrix}\cos b & \sin b\\-\sin b & \cos b\end{bmatrix}$$

These 3 different cases are 3 Jordan Canonical form for 2×2 matrices.

1. Diagonalizable Matrix (2 real & distinct)
2. non-diagonalizable matrix (real & equal)
3. complex conjugate matrix (complex conjugate)

3.5.2 Linear system of ODE & Exponential Matrix

Theorem The solution to the standard IVP $\vec{y}' = A\vec{y}$ with $\vec{y}(0) = \vec{y}_0$ and $A_{n\times n}$ = constant matrix (time-invariant), then

$$\vec{y} = e^{xA}\vec{y}_0$$

Proof

$$\begin{aligned}\vec{y} &= e^{xA}\vec{y}_0 \\ &= \left(I + xA + \frac{x^2A^2}{2!} + \dots\right)\vec{y}_0\end{aligned}$$

Differentiate both sides wrt x,

$$\frac{d\vec{y}}{dx} = \vec{y}' = \frac{d}{dx}\left(I + xA + \frac{x^2A^2}{2!} + \dots\right)\vec{y}_0$$

$$\vec{y}' = \left(A + \frac{x}{1!}A + \frac{x^2}{2!}A^3 + \dots\right)\vec{y}_0 = A\left(I + xA + \frac{x^2A^2}{2!} + \dots\right)\vec{y}_0$$

So $\vec{y}' = A\vec{y}$.

Ex Consider the system of ODEs

$$\vec{y}' = A\vec{y}, \quad A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with two distinct real roots.

Soln We want to find solution $\vec{y} = e^{xA}\vec{y}_0$, we have $e^{xA} = \begin{bmatrix} e^{ax} & 0 \\ 0 & e^{bx} \end{bmatrix}$.

If $\vec{y}(0) = \vec{y}_0$, then we have

$$\vec{y} = e^{xA}\underbrace{\vec{y}_0}_{\binom{a_1}{a_2}} = \begin{pmatrix} a_1e^{ax} \\ a_2e^{bx} \end{pmatrix}$$

This is a solution to linear system of ODEs with constant coefficients (real & distinct eigenvalues). Hence general solution is $\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 = c_1e^{ax} + c_2e^{bx}$

Ex2 Consider the system of ODEs,

$$\vec{y}' = A\vec{y} = A\vec{y}, \qquad A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \qquad \text{with real \& equal roots}$$

Soln We want to find (solution)

$$\vec{y} = e^{xA}\vec{y}_0$$

where $A = aI + N, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$e^{xA} = e^x(aI + N) = e^{ax}Ie^{xN} = e^{ax}I[I + xN] = e^{ax}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

The solution is

$$\vec{y} = e^{xA}\vec{y}_0 = e^{ax}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = e^{ax}\begin{pmatrix} a_1 + a_2x \\ a_2 \end{pmatrix}$$

For real single root of multiplicity 2, we obtain the G.S. as

$$\vec{y} = \begin{pmatrix} (a_1 + a_2x)e^{ax} \\ a_2e^{ax} \end{pmatrix}$$

Ex3 Consider $\vec{y}' = Ay, \quad A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

Soln Let $B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

$$e^{xA} = e^{x(aI+B)} = e^{ax} I e^{xB}$$
$$\vdots$$
$$\vec{y} = \ldots = \begin{pmatrix} e^{ax}(a_1 \cos(bx) + a_2 \sin(bx)) \\ e^{ax}(-a_1 \sin(bx) + a_2 \cos(bx)) \end{pmatrix}$$

We can extend this opposed to $n \times n$ matrices sing Eigenvalue decomposition (singularity transformation)

$$A = CBC^{-1}$$

where $B = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ for $\lambda_i \to$ real and distinct eigenvalues. C: consists of column vectors that are eigenvectors of the original matrix A.

$$C = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$$

where $\vec{v}_i$ is corresponding eigenvector for eigenvalue λ_i.

For a $2\times$ matrix, we write 2×2 block $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ if eigenvalues are repeated.

If eigenvalues are complex $\lambda_{1,2} = a \pm ib$, then Jordan Block is in the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$$\begin{aligned} e^{xA} &= e^{xCBC^{-1}} \\ &= I + xCBC^{-1} + \frac{x^2}{2!}(CBC^{-1})(CBC^{-1}) + \ldots \\ &= CIC^{-1} + xCBC^{-1} + \frac{x^2}{2!}(CBC^{-1})(CBC^{-1}) + \ldots \\ &= C\left[I + xB + \frac{x^2}{2}B^2 + \ldots\right] C^{-1} \\ &= Ce^{xB}C^{-1} \end{aligned}$$

Ex Consider $\vec{y}' = A\vec{y}, \quad A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

Soln $\lambda_1 = 3, \lambda_2 = -1$
$\vec{v}_1 = \binom{1}{2}, \vec{v}_2 = \binom{1}{-2}$

$$B = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

G.S

$$e^{xA} = Ce^{xB}C^{-1} = \ldots = \begin{bmatrix} \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} & \frac{1}{4}e^{3x} - \frac{1}{4}e^{-x} \\ e^{3x} - e^{-x} & \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} \end{bmatrix} \to \text{Fundamental matrix}$$

Ex Consider the vector DE $\vec{y}' = A\vec{y}, \quad A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

Soln To find eigenvalues, we put $\det(A - \lambda I) = 0 \implies \lambda = 2 \pm i$

$$\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad B = diag(\lambda_1, \lambda_2) = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$$

$$e^{xB} = \begin{bmatrix} e^{x(2+i)} & 0 \\ 0 & e^{x(2-i)} \end{bmatrix}, \; C = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}, \; C^{-1} = \begin{bmatrix} \frac{-1}{2i} & \frac{1}{2} \\ \frac{1}{2i} & \frac{1}{2} \end{bmatrix}$$

so

$$e^{xA} = Ce^{xB}C^{-1} = \ldots = e^{2x} \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

since $\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$

3.6 Non-homogeneous linear system of Equations

We start with

$$\frac{d\vec{y}}{dx} = A(x)\vec{y} + \vec{f}(x), \qquad \vec{y}(x_0) = \vec{y}_0$$

where $A(x)$ represents the internal dynamic of the system and $\vec{f}(x)$ represents the external force, which is responsible for inhomogeneity. The solution for the homogeneous part is the fundamental matrix.

Let's consider the scalar case

$$\frac{dy}{dx} = ay + f(x), \quad y(x_0) = y_0$$

$$\frac{dy}{dx} - ay = f(x), \qquad \text{Using IF, } I(x) = e^{ax}$$

Multiply by IF,

$$e^{-ax}\frac{dy}{dx} - ae^{-ay}y = e^{-ax}f(x)$$

$$\frac{d}{dx}(e^{-ax}) = e^{-ax}f(x)$$

Integrate then we get G.S. for scalar cases for any given IC,

$$y = y_0e^{a(x-x_0)} + \int_{x_0}^{x} e^{a(x-s)}f(s)ds$$

Consider the vector case, out solution to homogeneous part is

$$\vec{y}_h = \Phi(x, x_0)\vec{a}, \quad \forall \vec{a} \;\; (IC)$$

We can write our particular solution using variation of parameters as

$$\vec{y}_p = \Phi(x, x_0)\vec{v}(x)$$

Diff both sides,

$$\frac{d}{dx}\vec{y}_p = \vec{v}(x)\frac{d\Phi(x,x_0)}{dx} + \Phi(x,x_0)\frac{d\vec{y}}{dx} = A(x)\Phi(x,x_0) + \vec{f}(x)$$

Now $\frac{d\Phi(x,x_0)}{dx} = A\Phi(x,x_0)$

$$\implies A\Phi(x,x_0)\vec{v}_0 + \Phi(x,x_0)\frac{d\vec{v}}{dx} = A\Phi(x,x_0)\vec{v}_0 + \vec{f}(x) \implies \Phi(x,x_0)\frac{d\vec{v}}{dx} = \vec{f}(x)$$

By inverse Property,

$$\frac{d\vec{v}}{dx} = [\Phi(x,x_0)]^{-1}\vec{f}(x) = \Phi(x_0,x)\vec{f}(x)$$

$$\vec{v} = \int_{x_0} \Phi(x_0,s)\vec{f}(s)\ ds$$

$$\vec{y}_p = \Phi(x,x_0)\vec{v}(x) = \Phi(x,x_0)\int_{x_0}^{x} \Phi(x_0,s)\vec{f}(s)\ ds = \int_{x_0}^{x} \Phi(x,x_0)\Phi(x_0,s)\vec{f}(s)\ ds = \int_{x_0}^{x} \Phi(x,s)\vec{f}(s)\ ds$$

By multiplicative property

$$[x \to x_0, x_0 \to s \implies x \to s]$$

$\therefore$ GS of our inhomogeneous problem is

$$\vec{y} = \vec{y}_h + \vec{y}_p = \Phi(x,x_0)\vec{y}_0 + \int_{x_0}^{x} \Phi(x,s)\vec{f}(s)\ ds$$

Remark If system is time-invariant, the fundamental matrix is equal to exponential Matrix. So in this case

$$\Phi(x,x_0) = \Phi(x-x_0,0) = e^{A(x-x_0)}$$

So GS for linear system is

$$\vec{y} = e^{A(x-x_0)}\vec{y}_0 + \int_{x_0}^{x} e^{A(x-s)}\vec{f}(s)\ ds$$

Ex For time invariant system, we are given

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}(t)$$

where $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}, \vec{b}(t) = \begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix}$

Soln at $t_0 = 0$, we have

$$\vec{x}(t) = e^{At}\vec{x}_0 + \int_0^t e^{A(t-s)}\vec{v}(s)ds$$

We obtain eigenpairs $\lambda_1 = 3, \lambda_2 = -1,\ \vec{v}_1 = \binom{1}{2}, \vec{v}_2 = \binom{1}{-2}$

$$B = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$e^{xB} = \begin{bmatrix} e^{3x} & 0 \\ 0 & e^{-x} \end{bmatrix}$

$$e^{xA} = Ce^{xB}C^{-1} = \begin{bmatrix} \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} & \frac{1}{4}e^{3x} - \frac{1}{4}e^{-x} \\ \\ e^{3x} - e^{-x} & \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} \end{bmatrix}$$

we have

$$\begin{aligned} \vec{x}_p &= \int_0^t e^{A(t-s)}\vec{b}(s)ds \\ &= \int_0^t \begin{bmatrix} \frac{1}{2}e^{3(t-s)} + \frac{1}{2}e^{-(t-s)} & \frac{1}{4}e^{3(t-s)} - \frac{1}{4}e^{-(t-s)} \\ \\ e^{3(t-s)} - e^{-(t-s)} & \frac{1}{2}e^{3(t-s)} + \frac{1}{2}e^{-(t-s)} \end{bmatrix} \begin{pmatrix} e^{-2s} \\ 0 \end{pmatrix} ds \\ &= \dots \end{aligned}$$

$$\vec{x}(t) = \vec{x}_h + \vec{x}_p = \begin{bmatrix} \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} & \frac{1}{4}e^{3x} - \frac{1}{4}e^{-x} \\ \\ e^{3x} - e^{-x} & \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} \end{bmatrix} \vec{x}_0 + \frac{1}{10}e^{-2t}\begin{pmatrix} e^{5t} + 5e^t - 6 \\ 2e^{5t} - 2e^5 + 8 \end{pmatrix}$$

Ex (Previous Final Exam)
Given $A = \begin{bmatrix} p & 0 \\ s & q \end{bmatrix}$, find e^A, $p \neq q \neq 0, \quad p, q, s \in \mathbb{R}$.

Soln Let $\vec{x}' = A\vec{x}$

Step 1 (To find eigenpairs)

$$\det(A - \lambda I) = 0 \implies \lambda_1 = p, \lambda_2 = q$$

For $\lambda_1 = p \implies v_2 = \frac{s}{p-q}v_1 \implies v_1 = \binom{p-q}{s}$
For $\lambda_2 = q \implies \vec{v}_2 = \binom{0}{1}$

Step 2 G.S

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = \begin{pmatrix} c_1(p-q)e^{pt} \\ c_1 s e^{pt} + c_2 e^{qt} \end{pmatrix}$$

Step 3 If $x_0 = \binom{1}{0} = \binom{c_1(p-q)}{c_1 s + c_2} \implies c_1 = \frac{1}{p-q}, c_2 = \frac{s}{q-p}$
If $x_0 = \binom{0}{1} = \binom{c_1(p-q)}{c_1 s + c_2} \implies c_1 = 0, c_2 = 1$

Step 4 Fundamental Matrix

$$\Phi(t,0) = \begin{bmatrix} (p-q)e^{pt}\frac{1}{p-q} & 0 \\ \frac{1}{p-q} + \frac{s}{q-p}e^{qt} & e^{qt} \end{bmatrix}$$

Therefore

$$\Phi(t,0) = \begin{bmatrix} e^{pt} & 0 \\ \frac{s}{p-q}(e^{pt} - e^{qt}) & e^{qt} \end{bmatrix} = e^{At}$$

Step 5 Put $t = 1$,

$$e^A = \Phi(1,0) = \begin{bmatrix} e^p & 0 \\ \frac{s}{p-q}(e^p - e^q) & e^q \end{bmatrix}$$

CHAPTER 4

Laplace Transform Methods

LTM[1] provide important tools to solve Linear Order DEs and PDEs. It is especially useful when parts of the problem are discontinuous or non-differentiable.

Defn Given a real or complex valued function, $y(t)$, the LT, $\mathcal{L}$, of $y(t)$ is defined by

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} y(t) dt \qquad \forall s \in \mathbb{C}$$

such that the above improper integral converges.

Ex Given $f(t) = e^{ct}, c \neq 0$, $\mathcal{L}\{f(t)\} =?$

Soln

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{e^{ct}\right\} = \int_0^\infty e^{ct} e^{-st} dt = \lim_{b\to\infty} \int_0^b e^{(c-s)t} dt = \ldots = \frac{1}{s-c}$$

assume $\text{Re}(s) > \text{Re}(c)$.

$$\therefore \mathcal{L}\left\{e^{ct}\right\} = \frac{1}{s-c}, \quad \text{Re}(s) > \text{Re}(c)$$

If we have $f(t) = 1$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} dt = \lim_{b\to\infty} \int_0^b e^{-st} dt = \lim_{b\to\infty} -\left.\frac{e^{st}}{s}\right|_0^b = \frac{-1}{s} \lim_{b\to\infty} [e^{-sb} - 1] = \frac{-1}{s}[0-1] = \frac{1}{s}, \quad \text{Re}(s) > 0$$

Theorem If $f(t) = t^n, n \in \mathbb{N} = \{1, 2, \ldots, \}$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt = \frac{n!}{s^{n+1}}$$

[1] Laplace Transform Methods

Proof

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \int_0^\infty t^n e^{-st} dt \\
&= \lim_{b\to\infty} \int_0^b t^n e^{-st} dt \\
&= \lim_{b\to\infty} t^n \frac{e^{-st}}{-s}\Bigg|_0^b \nearrow 0 + \lim_{b\to\infty} \int_0^b n t^{n-1} \left(\frac{e^{-st}}{s}\right) dt \\
&= \frac{n}{s} \lim_{b\to\infty} \int_0^b t^{n-1} \left(e^{-st}\right) dt \\
&= \frac{n}{s} \lim_{b\to\infty} \int_0^b e^{-st} t^{n-1} dt \\
\mathcal{L}\{t^n\} &= \frac{n}{s} \mathcal{L}\left\{t^{n-1}\right\}
\end{aligned}$$

IBP repetitively,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \text{assume that } \operatorname{Re}(s) > 0$$

4.1 Properties of LT

1. Linearity: LT is a linear operator

$$\mathcal{L}\{c_1 f + c_2 g\} = c_1 \mathcal{L}\{f\} + c_2 \mathcal{L}\{g\}$$

   where c_1 & c_2 are arbitrary constants.

2. Existence: If $f(t)$ is a piecewise defined function at each interval $[0, b]$ for $b > 0$ and there exists a constant α such that

$$f(t) = O(e^{\alpha t})$$

   as $t \to \infty$, then $f(t)$ is said to be if exponential order α as $t \to \infty$. In other words, $F(s) = \mathcal{L}\{f(t)\}$ exists for $\operatorname{Re}(s) > \alpha$.

   Remark Big O notation is a convenient way to describe how fast a function is growing. Let $T(n)$ & $f(n)$ be 2 positive functions, we write $T(n) = O(f(n))$, and $T(n)$ has order of $f(n)$, there exists positive constants M and n_0 such that $T(n) \le M f(n)$ for all $n \ge n_0$.

3. Differentiation: If f is continuous and f' is piecewise continuous on any interval $[0, b]$ for $b > 0$ and f is of exponential order α as $t \to \infty$

$$\mathcal{L}\{f'\}\, s\mathcal{L}\{f\} - f(0) \qquad \text{for } \operatorname{Re}(s) > \alpha$$

Proof

$$\begin{aligned}
\mathcal{L}\{f'\} &= \int_0^\infty e^{-st} f'(t)dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st} f'(t)dt \\
&= \lim_{b\to\infty} e^{-st} f(t)\Big|_0^b - \lim_{b\to\infty} \int_0^b e^{-st}(-s)f(t)dt \\
&= (0 - f(0)) + s \lim_{b\to\infty} \left[\int_0^b e^{-st} f(t)dt\right] \\
&= -f(0) + sF(s) = sF(s) - f(0)
\end{aligned}$$

Similarly

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - s^0 f^{(n-1)}(0)$$

with the conditions that $f, f', f'', \ldots, f^{(n-1)}$ are all continuous and $f^{(n)}$ is piecewise continuous.

4. Shifting Properties:

   (a) 1^{st} shift theorem: (FST)
   If $F(s) = \mathcal{L}\{f(t)\}$ exists for $\text{Re}(s) > \alpha$ with $\alpha \geq 0$, then

   $$\mathcal{L}\left\{e^{ct} f\right\} = F(s-c) \qquad \text{for } \text{Re}(s-c) > \alpha$$

   where c is a constant

   Ex Find $\mathcal{L}\{t^3 e^{ct}\}$

   Soln

   $$\mathcal{L}\left\{t^3\right\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4} \qquad \text{for } \text{Re}(s) > 0$$

   So by FST,

   $$\mathcal{L}\left\{t^3 e^{ct}\right\} = F(s-c) = \frac{6}{(s-c)^4} \quad \text{for } \text{Re}(s-c) > 0$$

   Ex Find $\mathcal{L}\{t^5 e^{10t}\}$

   Soln $\mathcal{L}\{t^5\} = \frac{120}{s^6}$ $\text{Re}(s) > 0$.
   By FST,

   $$\mathcal{L}\left\{t^5 e^{10t}\right\} = \frac{120}{(s-10)^6}$$

   $\text{Re}(s-10) > 0$ or $s > 10$.

Heavy side Function and its LT

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \implies H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

$$\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}$$

for $s > 0, a > 0$ where a is a constant

(b) 2^{nd} Shift Theorem (SST)
If $F(s) = \mathcal{L}\{f(t)\}$ exists for $\text{Re}(s) > \alpha > 0$ and c is a positive constant, then

$$\mathcal{L}\{H(t-c)f(t-c)\} = e^{-ct}F(s) \qquad \text{for } \text{Re}(s) > \alpha \geq 0$$

Ex If $F(s) = \frac{p(s)}{q(s)}, q(s) \neq 0$. p and q are polynomial. We can use partial fraction decomposition / completing the square

Ex Find $\mathcal{L}^{-1}\left\{\frac{s}{s^2+5s+6}\right\}$ (PFD)

Soln

$$\frac{s}{s^2+5s+6} = \frac{3}{s+2} - \frac{2}{s+3}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+5s+6}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = 3e^{-2t} - 2e^{-3t}$$

Completing the square

Ex

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$$

Soln Completing the squares. To apply FST, we need to express everything in terms of $(s+2)$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} &= \mathcal{L}^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} \\ &= e^{-2t}\cos(t) - 2e^{-2t}\sin(t) \end{aligned}$$

4.2 Solving DEs with Laplace Transform

Ex Solve the following first order DE using LTM,

$$y' + ky = kA\cos(\omega t), \qquad y(0) = y_0$$

Soln Apply LT to DE and using linear properties/ICs. (For simplicity, w and ω are equivalent)

$$\mathcal{L}\{y' + ky\} = kA\mathcal{L}\{\cos(\omega t)\}$$
$$\mathcal{L}\{y'\} + k\mathcal{L}\{y\} = kA\left(\frac{s}{s^2+w^2}\right)$$

We define $\mathcal{L}\{y(t)\} = Y(s)$

$$sY(s) - y_0 + kY(s) = kA\left(\frac{s}{s^2+w^2}\right)$$

$$Y(s) = \frac{y_0}{s+k} + kA\left(\frac{s}{(s+k)(s^2+w^2)}\right)$$

$$\frac{s}{(s+k)(s^2+w^2)} = \frac{D}{s+k} + \frac{Es+F}{s^2+w^2} \implies \begin{cases} D = \frac{-k}{k^2+w^2} \\ E = \frac{k}{k^2+w^2} \\ F = \frac{w^2}{k^2+w^2} \end{cases}$$

$$Y(s) = \frac{y_0}{s+k} + \frac{kA}{k^2+w^2}\left(-k\frac{1}{s+k} + k\frac{s}{s^2+w^2} + w\frac{w}{s^2+w^2}\right)$$

Applying ILT, then

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = y_0\mathcal{L}^{-1}\left\{\frac{1}{s+k}\right\} + \frac{kA}{k^2+\omega^2}\left[k\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} - k\mathcal{L}^{-1}\left\{\frac{1}{s+k}\right\} + \omega\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}\right]$$

$$\implies y(t) = y_0e^{-kt} + \frac{kA}{k^2+\omega^2}[k(\cos(\omega t) - e^{-kt}) + \omega\sin(\omega t)]$$

Ex Solve the following SOLDE with constant coefficients by using LT:

$$y'' + 3y' + 2y = e^x, \qquad y(0) = 1, y'(0) = 2$$

Soln Taking LT and applying linearty property,

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^x\}$$

$$\implies [s^2 - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s-1}$$

Using ICs

$$[s^2Y(s) - s - 2] + 3[Y(s) - 1] + 2Y(s) = \frac{1}{s-1}$$

$$Y(s)[s^2 + 3s + 2] - (s+5) = \frac{1}{s-1}$$

$$\begin{aligned} Y(s) &= \frac{s+5}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)} \\ &= \frac{4}{s+1} - \frac{3}{s+2} + \frac{1}{6}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s+2} \\ &= \frac{7}{2}\frac{1}{s+1} - \frac{8}{3}\frac{1}{s+2} + \frac{1}{6}\frac{1}{s-1} \end{aligned}$$

Then

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{7}{2}e^{-x} - \frac{8}{3}e^{-2x} + \frac{1}{6}e^{x}$$

4.3 Convolution of Two functions

Consider the general case with non-homogeneous boundary conditions

$$y'' + py' + qy = u(t), \qquad y(0) = y_0, y'(0) = v_0$$

Taking LT and applying linear property,

$$\mathcal{L}\{y''\} + p\mathcal{L}\{y'\} + q\mathcal{L}\{y\} = \mathcal{L}\{u(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + p[sY(s) - y(0)] + qY(s) = U(s)$$

Applying ICs

$$Y(s) = \underbrace{\frac{(s+p)y_0 + v_0}{s^2+ps+q}}_{(i)} + \underbrace{\frac{U(s)}{s^2+ps+q}}_{(ii)}$$

Part (i) Homogeneous Problem and it can be solved using PFD[2] / completing the squares

y_h

Part (ii) represents the external force called forcing term. It represents a particular solution y_p. we have

$$\frac{U(s)}{s^2+ps+q} = U(s)G(s)$$

where $G(s) = \frac{1}{s^2+ps+q}$ is called Transfer Function.

$$Y(s) = Y_h(s) + Y_p(s)$$

$$y(t) = \mathcal{L}^{-1}\{Y_h(s)\} + \mathcal{L}^{-1}\{Y_p(s)\} = y_h + \mathcal{L}^{-1}\{U(s)G(s)\}$$

In general

$$\mathcal{L}\{fg\} \neq \mathcal{L}\{f\}\mathcal{L}\{g\}$$

we need convolution theorem to solve our particular solution.

We say that h is the convolution of 2 functions defined as $f * g$ where

$$h(t) = (f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

[2] partial fraction decomposition

4.3.1 Properties

1. Commutative
$$(f * g)(t) = (g * f)(t)$$

   Proof
$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$
   Change of variable, $u = t - \tau, \quad du = -d\tau$.
$$(f * g)(t) = \int_t^0 f(u)g(t-u)(-du) = \int_0^t g(t-u)f(u)du = (g * f)(t)$$

2. Distribution
$$f * (g_1 + g_2) = f * g_1 + f * g_2$$

3. Associativity
$$(f * g) * h = f * (g * h)$$

4. With zero is zero
$$f * 0 = 0 * f = 0$$

Ex $f(t) = t, g(t) = 1$. Calculate $f * g$.

Soln
$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$
$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{1\} = \frac{1}{s}$$
$$F(s)G(s) = \frac{1}{s^3}$$
$$t * 1 = \int_0^t (t-\tau)1 \; d\tau = \frac{t^2}{2}$$
we have
$$H(s) = \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{s^3} = F(s)G(s)$$
or
$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

4.3.2 Convolution Theorem

If $f(t)$ and $g(t)$ have LTs $F(s)$ $G(s)$ respectively for $\text{Re}(s) > \alpha$, then if we define $H(s) = F(s)G(s)$, we have

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = (f * g)(t) = (g * f)(t)$$

where $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

IVP $y'' + py' + qy = u(t)$, $y(0) = y_0, y'(0) = v_0$
We had

$$\begin{aligned} Y(s) &= G(s)U(s) + \frac{(s+p)y_0 + v_0}{s^2 + ps + q} \\ &= Y_p(s) + Y_h(s) \end{aligned}$$

Homo soln:

$$y_h(t) = \mathcal{L}^{-1}\left[\frac{(s+p)y_0 + v_0}{s^2 + ps + q}\right]$$

Non homo

$$y_p(t) = \mathcal{L}^{-1}\{Y_p(s)\} = (g * u)(t) = \int_0^t g(t-\tau)u(\tau)d\tau \qquad \text{where } g(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + ps + q}\right]$$

As an aside, it is worth mentioning that $y_h(t)$ solves the homogeneous DE with nonhomogeneous ICs whereas $y_p(t)$ solves the nonhomogeneous DE with homogeneous ICs.

Ex

$$y'' + y = u(t), \quad \text{with} \quad y(0) = y_0, \quad \text{and} \quad y'(0) = v_0$$

Soln See pg 116 of course note
and continued... If $u(t) = \sin(\omega t)$

$$y(t) = I + y_0\cos(t) + v_0\sin(t)$$

$$I = \frac{1}{2}\left(\frac{1}{1+\omega}(\sin\omega t + \sin(t)) - \frac{1}{\omega - 1}(\sin(\omega t) - \sin t)\right)$$

If $\omega \neq 1$,

$$y(t) = I + y_0\cos(t) + v_0\sin(t)$$

If $\omega = 1$, we are forcing the equation at the resonant frequency. There is no need to recalculate the integral. We can take the limit $\omega \to 1$ and apply L' Hopital's Rule as follows

$$\lim_{\omega \to 1}\int_0^t \sin(t-\tau)\sin(\omega t)d\tau = \frac{\sin(t) - t\cos t}{2}$$

In case of resonance, the solution grows linearly in time, for all time.

4.4 Linear System of FODEs

Consider the system of DEs (Time invariant systems)

$$\vec{x}' = A\vec{x} + \vec{f}(t)$$

with $\vec{x}(0) = \vec{x}_0$ where A is a constant matrix.

We complete LT of this vector DE

$$\mathcal{L}\{\vec{x}'\} = A\mathcal{L}\{\vec{x}\} + \mathcal{L}\left\{\vec{f}(t)\right\}$$

$$s\vec{X}(s) - \vec{x}_0 = A\vec{X}(s) + \vec{F}(s)$$

$$\vdots$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s)$$

The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is known as the transfer function matrix.

Theorem For any constant $n \times n$ matrix A,

$$\mathcal{L}\left[e^{t\mathbf{A}}\right] = (s\mathbf{I} - \mathbf{A})^{-1}$$

for values of s that satisfy $\text{Re}(s) > \text{Re}(\lambda)$ for all eigenvalues λ of A.

4.5 Laplace Transform of Heaviside (unit step) function

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

4.5.1 Properties

1. $\mathcal{L}\{H(t)\} = \frac{1}{s}$
2. $\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}, \quad a > 0$
3. second shift theorem/Time displacement Theorem, where

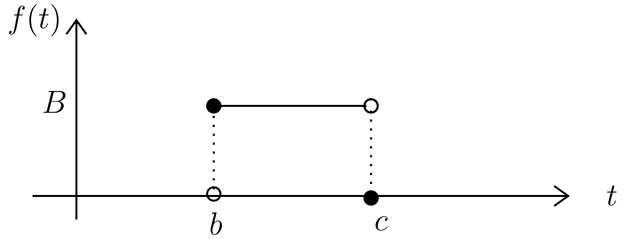
$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s), \quad F(s) = \mathcal{L}\{f(t)\}$$

Ex sketch the following function and obtain their LTs.

1. $f(t) = \begin{cases} 0 & t < b \\ B & b \leq t < c \\ 0 & t \geq c \end{cases}$

$$\implies f(t) = B[H(t-b) - H(t-c)]$$

b, c, B are positive constants

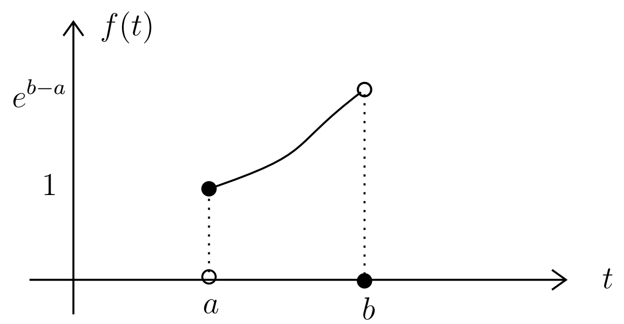


rectangular Pulse

$$\mathcal{L}\{f(t)\} = \frac{B}{s}(e^{-bs} - e^{-cs})$$

2. $f(t) = \begin{cases} 0 & t < a \\ e^{t-a} & a \leq t < b \\ 0 & t \geq b \end{cases}$

$0 < a < b$ constants.



$$f(t) = e^{t-a}[H(t-a) - H(t-b)]$$

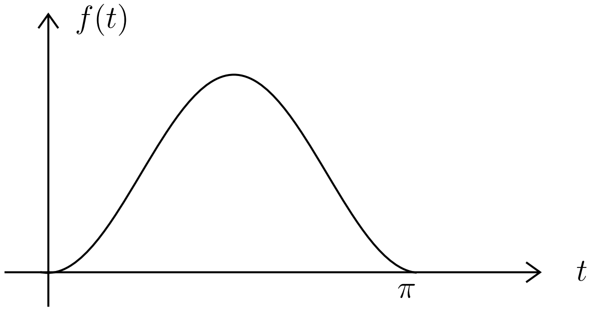
$$\mathcal{L}\{f(t)\} = \ldots = e^{-as}\frac{1}{s-1} - \mathcal{L}\left\{e^{t-a}H(t-b)\right\}$$

$a \neq b$ arguments is diff.

$$t - a = (b-a) + (t-b)$$
$$e^{t-a} = e^{b-a}e^{t-b}$$

$$\begin{aligned}\mathcal{L}\left\{f(t)\right\} &= e^{-as}\frac{1}{s-1} - e^{b-a}\mathcal{L}\left\{H(t-b)e^{t-b}\right\} \\ &= e^{-as}\frac{1}{s-1} - e^{b-a}\frac{e^{-bs}}{s-1} \\ &= \text{final answer}\end{aligned}$$

3. $f(t) = \begin{cases} 0 & t < 0 \\ \sin t & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$



$$f(t) = \sin(t)[\underbrace{H(t)}_{start} - \underbrace{H(t-\pi)}_{end}]$$

$$\begin{aligned}\mathcal{L}\left\{f(t)\right\} &= \mathcal{L}\left\{\sin(t)H(t)\right\} - \mathcal{L}\left\{\sin(t)H(t-\pi)\right\} \\ &= \mathcal{L}\left\{\sin(t)H(t)\right\} + \mathcal{L}\left\{\sin(t-\pi)H(t-\pi)\right\} \\ &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} \\ &= \frac{1+e^{-\pi s}}{s^2+1}\end{aligned}$$

4.6 Dirac Deltas Functions & its LT

Heaviside Function is used for switching states. We use Dirac Delta Function when a large force acts over a small period of time (e.g. high pulsed impulse, hammering)

4.6.1 Properties of Dirac Delta Function

1.

$$\delta(t-a) = 0, \quad t \neq a$$

2.

$$\int_{a-\epsilon}^{a+\epsilon} \delta(t-a)dt = 1$$

or

$$\int_{-\infty}^{+\infty} \delta(t-a)dt = 1$$

e.g.

$$\int_1^3 \delta(t-5)dt = 0$$

since $5 \notin [1,3]$

$$\int_1^3 \delta(t-2)dt = 1$$

since $2 \in [1,3]$

3.

$$\int_{a-\epsilon}^{a+\epsilon} f(t)\delta(t-a)dt = f(a)$$

or

$$\int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = f(a)$$

4.

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-as} \qquad a \geq 0$$

Ex Solve the IVP

$$y'' + 2y' - 15y = 6\delta(t-9) \qquad y(0) = -5,\ y'(0) = 7$$

Soln Apply LT, using linearity property, and ICs.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 15\mathcal{L}\{y\} = 6\mathcal{L}\{\delta(t-9)\}$$
$$\ldots = 6e^{-9s}$$
$$\implies Y(s) = \frac{6e^{-9s}}{s^2+2s-15} - \frac{5s+3}{s^2+2s-15} = 6e^{-9s}F(s) - G(s)$$
$$Y(s) = 6e^{-9s}\left(\frac{1/8}{s-3} - \frac{1/8}{s+5}\right) - \text{using partial fractions}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t}$$
$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{9}{4}e^{3t} + \frac{11}{4}e^{-5t}$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \text{shift theorem} = 6H(t-9)f(t-9) - g(t)$$

4.6.2 Relation b/w Dirac Delta Function and Heaviside (Unit Step) Function

$$\int_{-\infty}^{\infty} \delta(u-a)du = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} = H(t-a)$$

By Fundamental thm of calculus

$$\frac{d}{dt}H(t-a) = \frac{d}{dt}\int_{-\infty}^{\infty} \delta(u-a)du = \delta(t-a)$$

Ex Solve the IVP (Previous Final Exam)

$$2y'' + 10y = 3H(t-12) - 5\delta(t-4) \qquad \text{with } y(0) = -1,\ y'(0) = -2$$

Soln We apply LT, linearity Property, and use ICs,

$$2\mathcal{L}\{y''\} + 10\mathcal{L}\{y\} = 3\mathcal{L}\{H(t-12)\} - 5\mathcal{L}\{\delta(t-4)\}$$

$$2(s^2Y(s) - sy(0) - y'(0)) + 10Y(s) = 3\frac{e^{-12s}}{s} - 5e^{-4s}$$

$$Y(s) = e^{-12s}\underbrace{\frac{1}{s(2s^2+10)}}_{F(s)} - 5e^{-4s}\underbrace{\frac{1}{2s^2+10}}_{G(s)} - \underbrace{\frac{2s+4}{2s^2+10}}_{H(s)}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{10} - \frac{1}{10}\cos\left(\sqrt{5}t\right)$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{2\sqrt{5}}\sin\left(\sqrt{5}t\right)$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \cos\left(\sqrt{5}t\right) + \frac{2}{\sqrt{5}}\sin\left(\sqrt{5}t\right)$$

$$y(t) = 3H(t-12)f(t-12) - 5H(t-4)g(t-4) - h(t)$$

4.7 Periodic application of Dirac Delta Function

(pg 128-130) Suppose there is a radioactive material in a container that decays at a rate of k. x_0 is the initial concentration of the radioactive material. We can describe it as IVP

$$\frac{dx}{dt} = -kx, \quad x(0) = x_0$$

with solution

$$x(t) = x_0e^{-kt} \qquad \text{(vanishes at } t = \infty)$$

case 2 We add A amount of the same material at any time $t = a$. Then at $t = a$, we have $x(t) = x_0e^{-ka} + A$. In terms of Heaviside Function

$$x(t) = \begin{cases} x_0e^{-kt} & 0 \le t < a \\ (x_0e^{-ka} + A)e^{-k(t-a)} & t \ge a \end{cases}$$

$$x(t) = x_0e^{-kt} + Ae^{-k(t-a)}H(t-a)$$

we have

$$\frac{d}{dt}x(t) = -kx(t) + f(t)$$

where $f(t) = A\delta(t-a)$ describes the instantaneous addition of A to the material at $t = a$.

case 3 We add A amount of same material periodically at every time T (units of time) we write

$$\frac{dx}{dt} = -kx + f(t)$$

where $f(t) = A\sum_{k=1}^{\infty} \delta(t - nT)$

We take LT,

$$(sX(s) - x(0)) + kX(s) = F(s)$$
$$X(s) = \frac{x_0}{s+k} + \frac{F(s)}{s+k}$$

Inverse LT,

$$x(t) = \underbrace{x_0 e^{-kt}}_{=x_h} + \underbrace{(g * f)(t)}_{x_p}$$

where $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+k}\right\} = e^{-kt}$
$x_h \to 0$ as $t \to \infty$

Remark ICs don't make any contribution towards long-time behaviour.

$$\begin{aligned}
x_p(t) = (g * f)(t) &= \int_0^t f(t-\tau) f(\tau) d\tau \\
&= A \sum_{n=1}^{\infty} \int_0^t e^{k\tau} \delta(\tau - nR) d\tau \\
&= Ae^{-kt} \sum_{n=1}^{\infty} \int_0^t e^{k\tau} \delta(\tau - nT) d\tau \qquad \text{Dirac Delta (prop. 3)} \\
&= Ae^{-kt} \sum_{n=1}^{\infty} e^{knT} \\
&= Ae^{-kt}\left[e^{kT} + e^{2kT} + \ldots + e^{NkT}\right] \\
&= Ae^{k(T-t)}\left[1 + e^{kT} + e^{2kT} + \ldots + e^{(N-1)kT}\right]
\end{aligned}$$

Assuming $t = NT + u, \quad 0 \leq u < T$

$$\begin{aligned}
x_p &= Ae^{k(-(N-1)T-u)}[1 + e^{kT} + \ldots + e^{(N-1)kT}] \\
&= Ae^{-ku}[1 + e^{-kT} + \ldots + e^{-(N-1)kT}]
\end{aligned}$$

In the limit $N \to \infty, (t \to \infty)$,

$$x_p = \frac{Ae^{-ku}}{1 - e^{-kT}} \qquad \text{long time behaviour}$$

CHAPTER 5

Perturbation Methods (Theory)

Very few DEs are exactly solvable. So we need approximate methods.

Approximate Methods

1. Numerical Methods
2. Perturbation Theory (Method)

We introduce a small parameter ϵ in the given equation by writing approximate solution in term of ϵ. If we take $N(steps) \to \infty$, then Perturbation Solution will converge to exact solution. Our solution will have leading term (0^{th} order), first order and second order correction. 3 different techniques

- Regular PT
- Singular PT
- Poincare PT

5.1 Regular PT

(to find roots of algebraic Equation)

Consider algebraic equation

$$x^2 + (y+\epsilon)x + (3-2\epsilon) = 0 \tag{5.1}$$

Roots of quadratic are

$$x_{\pm} = \frac{-(y+\epsilon) \pm \sqrt{(y+\epsilon)^2 - 12 + 8\epsilon}}{2} \tag{5.2}$$

Assuming $\epsilon \ll 1$, we write our approximate solution as

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{5.3}$$

sub (5.3) into (5.1)

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots)^2 + (y + \epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) + (3 - 2\epsilon) = 0$$

$$(x_0^2 + yx_0 + 3)\epsilon^0 + (2x_0x_1 + yx_1 + x_0 - 2)\epsilon^1 + (x_1^2 + 2x_0x_2 + yx_2 + x_1)\epsilon^2 + \dots = 0$$

compare the coefficients

$$\epsilon^0 : \qquad x_0^2 + yx_0 + 3 = 0 \implies x_0 = -3, -1$$

These are roots of given equation by subbing $\epsilon = 0$.

$$\epsilon^1 : \qquad 2x_0x_1 + yx_1 + x_0 - 2 = 0 \implies x_1 = \frac{2 - x_0}{2x_0 + y}$$

For

$$\begin{array}{lll} x_0 = -3 & x_1 = -5/2 & x_2 = 15/8 \\ x_0 = -1 & x_1 = 3/2 & x_2 = -15/8 \end{array}$$

similarly, to find x_2,

$$\epsilon^2 : \qquad x_1^2 + 2x_0x_2 + 4x_2 + x_1 = 0 \implies x_2 = \frac{-x_1^2 - x_1}{2x_0 + 4}$$

We obtain 2 approximate solution as

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \implies \begin{array}{l} x = -3 - \frac{5}{2}\epsilon + \frac{15}{8}\epsilon^2 + \dots \\ \\ x = -1 + \frac{3}{2}\epsilon - \frac{15}{8}\epsilon^2 + \dots \end{array}$$

To find the exact solution, we use linear approximation

$$\sqrt{(4+\epsilon)^2 - 12 + 8\epsilon} = 2\sqrt{1 + 4\epsilon + \frac{\epsilon^2}{4}} \approx 2\sqrt{1 + 4\epsilon}$$

($\frac{\epsilon^2}{4}$ being very small, neglected)
(If $|x| < 1 \implies (1+x)^n = 1 + nx$ Binomial expansion)

$$\implies x_{\pm} = \frac{-4 - \epsilon \pm (2 + 4\epsilon)}{2} = \begin{cases} -1 + \frac{3}{2}\epsilon \\ \\ -3 - \frac{5}{2}\epsilon \end{cases}$$

PT is acceptable

5.2 Singular Perturbation Method

Consider equation

$$\epsilon x^2 + 2x - 1 = 0 \tag{5.1}$$

Roots of quadratic equation are

$$x_{\pm} = \frac{-2 \pm \sqrt{4+4\epsilon}}{2\epsilon}$$

We propose approximate solution as ($\epsilon \ll 1$)

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{5.2}$$

sub (5.2) into (5.1)

$$\implies \dots \implies (2x_0 - 1)\epsilon^0 + (x_0^2 + 2x_1)\epsilon^1 + \epsilon^2(2x_0x_1 + 2x_2) = 0$$

$$\begin{array}{lll} \epsilon^0: & 2x_0 = 1 \implies x_0 = 1/2 & \text{heading order correction} \\ \epsilon^1: & x_0^2 + 2x_1 = 0 \implies x_1 = -1/8 & \text{first order correction} \\ \epsilon^2: & 2x_0x_1 + 2x_2 = 0 \implies x_2 = 1/16 & \text{second-order correction} \end{array}$$

Our approximate solution is

$$x = \frac{1}{2} - \frac{1}{8}\epsilon + \frac{1}{16}\epsilon^2 + \dots$$

This gives us only one solution using Regular PT because the equation is singular. We need to rescale the original equation.

Let

$$X = x\epsilon^{\nu} \text{ as } \epsilon \to 0 \tag{5.3}$$

$$x = \frac{X}{\epsilon^{\nu}}$$

sub this into (5.1), we obtain

$$\epsilon^{1-2\nu}X^2 + 2X\epsilon^{-\nu} - 1 = 0 \tag{5.4}$$

We want to balance the magnitude of first term ($\epsilon^{1-2\nu}X^2$) and the second term ($+2X\epsilon^{-\nu}$) to obtain a solution for quadratic equation

$$\implies 1 - 2\nu = -\nu \implies \nu = 1$$

so $x = \frac{X}{\epsilon}$

Eq (5.4) becomes

$$X^2 + 2X - \epsilon = 0 \tag{5.5}$$

We propose approximate solution as

$$X = X_0 - \epsilon X_1 + \epsilon^2 X_2 + \dots \tag{5.6}$$

Sub (5.6) into (5.5), we obtain

$$(X_1 + \epsilon X_1 + \epsilon^2 x_2 + \dots)^2 + 2(X_1 + \epsilon X_1 + \epsilon^2 x_2 + \dots) - \epsilon = 0$$

$$(X_0^2 + 2X_0) + \epsilon^1(2X_0X_1 + 2X_1 - 1) + \epsilon^2(\ ) + \ldots = 0$$
$$\epsilon^0 : \text{coeff} = 0 \implies X_0 = 0, -2$$
$$\epsilon^1 : \text{coeff} = 0 \implies X_1 = \frac{1}{2}, \frac{-1}{2}$$
$$\implies \begin{cases} X = -2 - \frac{1}{2}\epsilon + \ldots \\ X = 0 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \ldots \end{cases}$$

$x = \frac{X}{\epsilon}$

$$\begin{cases} X = -2 - \frac{1}{2}\epsilon + \ldots & x = \frac{1}{2} - \frac{1}{8}\epsilon + \frac{1}{16}\epsilon^2 + \ldots \\ X = 0 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \ldots & x = \frac{-2}{\epsilon} - \frac{1}{2} + \ldots \end{cases}$$

already found the first solution using RPT. We have assumed that $|x_0| > |x_1| > |x_2$.

5.3 Example

Ex Given

$$f(x,\epsilon) = x^2 - 1 + \epsilon = 0 \tag{5.1}$$

ⓐ apply RPT to find correction to second order by using

$$x(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + \ldots \tag{5.2}$$

ⓑ The exact roots of (5.1) are know i.e.

$$x^{(1)} = \sqrt{1-\epsilon}, \quad x^{(2)} = -\sqrt{1-\epsilon}$$

Show that perturbation roots from (a) agree with Taylor series expansion for $x^{(i)}$ to order ϵ^2. $(i = 1, 2)$

(a) Given (5.1) with unperturbed roots, $f(x,0) = 0 \implies x = \pm 1 \quad (2)$. Approx solution using the form (5.2). Sub 5.2 into 5.1,

$$(X_0 + \epsilon X_1 + \epsilon^2 X_2 + \ldots)$$
$$\implies \ldots$$

collecting powers of ϵ^n

$$\begin{cases} X_0 = \pm 1 \text{ as expected} \\ X_1 = \frac{-1}{2X_0} \\ X_2 = -\frac{X_1^2}{2X_0} = -\frac{1}{8X_0^3} \end{cases}$$

- for $X_0 = 1, X_1 = -\frac{1}{2}, X_2 = -\frac{1}{8}$

$$x^{(1)} = 1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3)$$

- $X_0 = -1, X_1 = \frac{1}{2}, X_2 = \frac{1}{8}$

$$X^{(2)} = -1 + \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + O(\epsilon^3)$$

(b) Taylor Series expansion of $g(\epsilon) = \sqrt{1-\epsilon}$

$g(\epsilon) = (1-\epsilon)^{\frac{1}{2}}$ about $\epsilon = 0$ to $O(\epsilon^2)$

$$g(0) = 1, g'(0) = -\frac{1}{2}, g''(0) = -\frac{1}{4}$$

$$x^{(1)} = g(0) + g'(0)\epsilon + \frac{g''(0)}{2!}\epsilon^2 + O(\epsilon^3)$$

agree with the previous result.

Similarly to $x^{(2)}$

5.4 Definition of Order Symbol $O, o, \sim$

Definition 7.2.1 (Big Oh). *Let $f(x)$ and $g(x)$ be two functions defined in some interval around x_0. We say that f is big-oh of g, or mathematically*

$$f(x) = O(g(x)), \quad \text{as } x \to x_0,$$

if

$$\lim_{x \to x_0} \left(\left| \frac{f(x)}{g(x)} \right| \right) = C < \infty.$$

That is to say the Taylor expansions of the two functions about x_0 both have the same first non-zero term, but could have different coefficients.

eg $f(x) = \sin(3x), g(x) = x \implies f(x) = O(g(x))$ as $x \to 0$ because $\lim_{x\to 0} \left|\frac{f(x)}{g(x)}\right| = 3 < \infty$

$$\sin(3x) = O(x)$$

Definition 7.2.2 (little oh). *Let $f(x)$ and $g(x)$ be two functions defined in some interval around x_0. We say that f is little-oh of g, or mathematically*

$$f(x) = o(g(x)), \quad \text{as } x \to x_0,$$

if

$$\lim_{x \to x_0} \left(\left| \frac{f(x)}{g(x)} \right| \right) = 0.$$

That is to say the first non-zero coefficient in the Taylor expansion of f about x_0 appears later on than for g.

eg $f(x) = \sin(x), g(x) = \sqrt{x} \implies f(x) = o(g(x))$ as $x \to 0$.

Definition 7.2.3 (Similar). *Let $f(x)$ and $g(x)$ be two functions defined in some interval around x_0. We say that f is similar to g, or mathematically*

$$f(x) \sim g(x), \quad \text{as } x \to x_0,$$

if

$$\lim_{x \to x_0} \left(\left| \frac{f(x)}{g(x)} \right| \right) = 1.$$

That is to say the Taylor expansions of the two functions about x_0 both have the same first non-zero term and their coefficients are equal.

eg $f(x) = \sin x, g(x) = x$ as $x \to 0$, $f(x) \sim g(x)$ by defn

5.5 Duffing's Equation

The non linear equation of pendulum is given by

$$ML\frac{d^2\theta}{dt^2} = -Mg\sin\theta$$

where L is the length of the pendulum, M is the mass of the bob, g is the force of gravity, t is time and θ is the angle that the pendulum makes with the vertical. Let's assume

$$u = \frac{\theta}{\epsilon} \quad \text{and} \quad t = \sqrt{\frac{L}{g}}x$$

then

$$\frac{d^2u}{dx^2} = -\frac{\sin(\epsilon u)}{\epsilon}$$

with

$$u(0) = 1, \quad \text{and} \quad u'(0) = 0$$

Using Taylor's expansion

$$\frac{\sin(\epsilon u)}{\epsilon} = u - \frac{\epsilon^2}{6}u^3 + O\left(\epsilon^4\right)$$

redefining $\epsilon = -\epsilon^2/6$. Then we get Duffing's equation in terms of t

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0 \tag{5.1}$$

with $\epsilon \ll 1$, $u(0) = 1$, and $u'(0) = 0$.

We propose approximate solution as

$$u(t, \epsilon) = u_0(t) + \epsilon u_1(t) + \ldots \tag{5.2}$$

$\epsilon \ll 1, u_0 > u_1 > u_2 \ldots$.

sub 5.2 into 5.1

$$\frac{d^2}{dt^2}(u_0(t) + \epsilon u_1(t) + \ldots) + (u_0(t) + \epsilon u_1(t) + \ldots) + \epsilon(u_0(t) + \epsilon u_1(t) + \ldots)^3 = 0$$

$$\left[\frac{d^2u_0}{dt^2} + u_0\right] + \left[\frac{d^2u_1}{dt^2} + u_1 + u_0^3\right]\epsilon + O\left(\epsilon^2\right) = 0$$

with Boundary conditions

$$u_0(0) + \epsilon u_1(0) + \cdots = 1$$
$$d\tfrac{du_0}{dt}(0) + \epsilon\frac{du_1}{dt}(0) + \cdots = 0$$

leading term ϵ^0: $u_0(t) = \cos(t)$
High orders ϵ^1: $\frac{d^2u_1}{dt^2} + u_1 = -u_0^3$. To solve it use laplace transform. see course note for details.

PT solution to Duffing's Equation is

$$u(t) = \cos(t) + \epsilon[-\frac{3}{8}\underbrace{t\sin t}_{\text{secular term}} + \frac{1}{32}(\cos 3t - \cos t)]$$

here $u_1(t) \gg u_0(t)$ as $t \to \infty$ Not allowed since $u_0 > u_1 > u_2 \ldots$.

The reason why this appears is that the forcing term in u_1 is resonant because the natural frequency of the leading term and next higher order term are same. $\to$ solution is Poincare Method.

5.6 Poincare Method

$$\frac{d^2u}{dt^2}+u+\epsilon u^3=0, \quad u(0)=1, \frac{du(0)}{dt}=0$$

Here RPT approach failed. We propose a new solution of the form (to get rid of secular term)

$$t=\left(1+\epsilon\omega_1+\epsilon^2\omega_2+\cdots\right)\tau$$

We have introduced new parameter τ w_1 and w_2 are unknown parameter (to be determined)

$$\begin{aligned}\frac{d}{dt}&=\frac{d\tau}{dt}\frac{d}{d\tau}=\left(1+\epsilon\omega_1+\epsilon^2\omega_2\right)^{-1}\frac{d}{d\tau}\\ \frac{d^2}{dt^2}&=\left(\frac{d\tau}{dt}\right)^2\frac{d^2}{d\tau^2}=\left(1+\epsilon\omega_1+\epsilon^2\omega_2\right)^{-2}\frac{d^2}{d\tau^2}\end{aligned}$$

$$\Longrightarrow \begin{array}{c}\frac{d^2u}{d\tau^2}+\left(1+\epsilon\omega_1+\epsilon^2\omega_2\right)^2\left(u+\epsilon u^3\right)=0\\ \frac{d^2u}{d\tau^2}+\left(1+2\epsilon\omega_1+\epsilon^2\left(\omega_1^2+2\omega_1\right)\right)\left(u+\epsilon u^3\right)=0\end{array}$$

We propose

$$u=u_0(\tau)+\epsilon u_1(\tau)+\epsilon^2u_2(\tau)+O\left(\epsilon^2\right)$$

and resulting equations is

$$\frac{d^2}{d\tau^2}\left(u_0+\epsilon u_1+\epsilon^2u_2\right)+\left(1+2\epsilon\omega_1+\epsilon^2\left(\omega_1^2+2\omega_1\right)\right)\left(u_0+\epsilon u_1+\epsilon^2u_2+\epsilon\left[u_0^3+3\epsilon u_0^2u_1\right]\right)=O\left(\epsilon^3\right)$$

We can then write down the equations for the first three orders

$$\begin{aligned}&\frac{d^2u_0}{d\tau^2}+u_0=0\\ &\frac{d^2u_1}{d\tau^2}+u_1=-u_0^3-2\omega_1u_0\\ &\frac{d^2u_2}{d\tau^2}+u_2=-3u_0^2u_1-2\omega_1\left(u_1+u_0^3\right)-\left(\omega_1^2+2\omega_2\right)u_0\end{aligned}$$

The ICs yield,

$$\begin{aligned}u_0(0)=1, &\quad \frac{du_0}{d\tau}(0)=0\\ u_1(0)=0, &\quad \frac{du_1}{d\tau}(0)=0\\ u_2(0)=0, &\quad \frac{du_2}{d\tau}(0)=0\end{aligned}$$

$u_0(\tau)=\cos(\tau)$ and u_1:

$$\begin{aligned}\frac{d^2u_1}{d\tau^2}+u_1&=-u_0^3-2\omega_1u_0\\ &=-\cos^3\tau-2\omega_1\cos\tau\\ &=\left(-2\omega_1-\frac{3}{4}\right)\cos\tau-\frac{1}{4}\cos 3\tau\end{aligned}$$

The first term on RHS represents secular term. We avoid this term by plugging coeff of $\cos(\tau)=0$

$$\Longrightarrow -2w_1-\frac{3}{4}=0 \Longrightarrow w_1=-\frac{3}{8} \Longrightarrow \frac{d^2u_1}{d\tau^2}+u_1=-\frac{1}{4}\cos(3\tau)$$

The homogeneous solution is $u_{1h} = A\cos\tau + B\sin\tau$ and the particular solution is of the form $u_{1p} = C\cos 3\tau$. By substituting into the equation, $C = \frac{1}{32}$. Therefore, our complex solution at order ϵ is

$$u_1 = A\cos\tau + B\sin\tau + \frac{1}{32}\cos 3\tau$$

To satisfy the zero ICs we need $A = \frac{-1}{32}, B = 0$, to yield

$$u_1 = \frac{1}{32}(\cos 3\tau - \cos\tau)$$

PT solution (upto first order, 2 first term correction)

$$u = \cos\tau + \frac{\epsilon}{32}(\cos 3\tau - \cos\tau)$$

with

$$t = \left(1 + \frac{3\epsilon}{8} - \frac{95}{1024}\epsilon^2\right)\tau$$