AMATH 353

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Contents

CHAPTER 1

May 6

- Definition of PDE, examples, terminology
- 1D conservation law

Defn A partial differential equation is an equation that relates to an unknown function u and its partial derivatives (u is a function of 2 or more variables)

Example $u = u(x, y) \leftrightarrow u$ is a function of x, y.
$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ $\quad u = \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)^2$ $\quad \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x^2 + y^2$

1.1 Learning Objectives

- Solve PDEs
- Model physical processes using PDEs

PDEs are very different than ODEs

$\frac{\partial u}{\partial v} = 0 \iff u$ is independent of $x \rightarrow u(x, y) = f(y)$ (most general solution)

Consequences Unlike ODEs, if we specify u at a single point x_0, y_0, we don't fully determine the solution

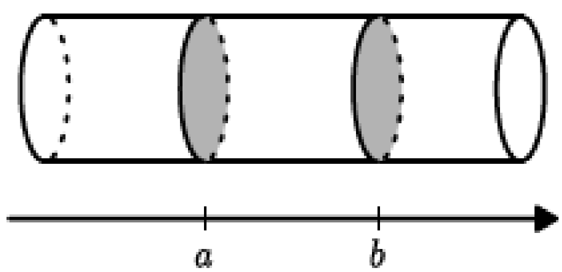
Linear unknown and its partial derivative appear alone and to the first power.

homogeneous A LINEAR PDE is called homogeneous if every term contains the unknown or one of its derivatives.

1.2 Modelling

- most of the time, u depends on time (t)
  evolution of a process

Transport equations some quantity is transported 1D



some element of a physical quantity is present in the cylinder.
Let $Q(t)$ be the amount of this quantity at time t in the section of cylinder with $a \leq x \leq b$. Want to describe $\frac{dQ}{dt}$ (rate of change)

$$\frac{dQ}{dt} = \text{ net rate at which } Q \text{ changes}$$

Assumptions

1. $Q(t)$ can be locally described by a density $\rho(x, y, z, t)$ with ρ continuous
   note that
   $$[\rho] = \frac{[Q]}{\text{volume}}$$
2. ρ is constant on cross-sections $\rightarrow \rho = \rho(x, t)$ only.
   makes sense if cylinder is insulated

model $\neq$ reality

$$Q(t) = \int_a^b \rho(x, t) A \; dx$$

where A is the cross-sectional area
ρ is local
Need to describe the rate of change

1.3 Other

- syllabus (next time)
- Tutorials
  - Done by me
  - no catching up with lectures
  - fun interesting different

CHAPTER 2

May 8

total amount $Q(t) = \int_a^b \rho(x,t)A\ dx$
$\frac{dQ}{dt}$ =? How can $Q(t)$ change?

1. quantity enters or exits from cross-sections $x = a, x = b$
2. quantity created/destroyed by an external source

Assumptions

1. assume that transport is locally described by flux

   $\phi(x,t) \to$ measurs amount of quantity crossing the cross secion at x at time t, per unit time, per unit area.

   $$[\phi] = \frac{[Q]}{area\ \cdot\ time} \implies$$

   net rate of change due to transport: $T(t) = \underbrace{\phi(a,t)}_{amount\ entering\ left} A - \underbrace{\phi(b,t)}_{amount\ entering\ right} A$

   Convention $\phi(x,t) \begin{cases} > 0 & \text{if moving right} \\ < 0 & \text{if moving left} \end{cases}$

2. Assume that quantity creation is locally described by source term $f(x,t,\rho)$

   measures quantity created/destroyed per unit volume, per unit time

   $$[f] = \frac{[Q]}{volume\ \cdot\ time} \to$$

   net rate of creation/destruction is

   $$S(t) = \int_a^b f(x,t,\rho(x,t))A\ dx$$

Convention $f(x,t,\rho)\begin{cases} > 0 & \text{if creation} \\ < 0 & \text{if destruction} \end{cases}$

$$\frac{dQ}{dt} = T(t) + S(t)$$

$$\to \frac{d}{dt}\int_a^b \rho(x,t)A dx = \phi(a,t)A - \phi(b,t)A + \int_a^b f(x,t,\rho(x,t))A dx$$

Global conservation law

Extra Assumptions $\frac{\partial \rho}{\partial t}$ and $\frac{\partial \rho}{\partial x}$ are continuous

fundamental theorem of calculus

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = \int_a^b \frac{\partial \rho}{\partial t}dx$$

$$\int_a^b \frac{\partial \phi}{\partial x}(x,t)dx = \phi(b,t) - \phi(a,t)$$

$$\int_a^b \left(\frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} - f\right) dx = 0$$

does this mean that $\frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} = f$?

Note that a, b are arbitrary

CHAPTER 3

May 10

$$\int_a^b \left(\frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} - f(x, y, \rho(x,t)) \right) dx = 0$$

The equation above is GLOBAL

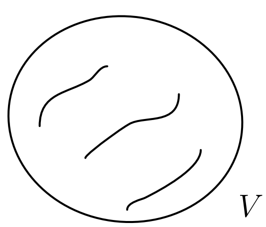
- a, b are arbitrary
- if we assume integrand continuous

$$\implies \frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} = f(x, t, \rho)$$

  local conservation law

  # of unknowns?
  We assume that f is known, therefore 2 unknowns.

To have any hope of solving, need to relate ρ and ϕ $\rightarrow$ constitutive relation



$\rightarrow$ arbitrary "nice" region of space

∂V = boundary of V (surface)

$Q(t)$ total amount of some quantity in V.
Assume

$$Q(t) = \iiint_V \rho(\vec{r}, t) dV$$

where $\vec{r} = (x, y, z)$

$$\frac{dQ}{dt} = \underbrace{T(t)}_{\text{net rate of change due to transport}} + \underbrace{S(t)}_{\text{net rate of change due to sources}}$$

1. $T(t)$ locally described by flux $\vec{\phi}(\vec{r},t)$
   - direction of $\vec{\phi}$: in which direction quantity flows
   - magnitude of $\vec{\phi}$: how much quantity is transported per unit time, per unit area (cross-section area, $\perp$ to $\vec{\phi}$)

$$[\vec{\phi}] = \frac{[Q]}{time \ \cdot \ area}$$

$$T(t) = -\iint_{\partial V} \vec{\phi}(\vec{r},t)\cdot\vec{n}(\vec{r})dA \qquad \text{surface integral}$$

   Net change

   - quantity either enters or exists from the boundary
     $\vec{n}(\vec{r})$ unit outward normal to ∂V

   minus sign $T > 0$ if $\vec{\phi}$ goes in

2. sources

$$S(t) = \iiint_V \underbrace{f(\vec{r},t,\rho(\vec{r},t))}_{source \ term} dV$$

3.1 Global Conservation Law

$$\frac{d}{dt}\iiint_V \rho \ dV = -\iint_{\partial V}\vec{\phi}\cdot\vec{n} \ dA + \iiint_V f \ dV$$

We assume $\rho, \vec{\phi}$ and f are "nice"

- we can apply theorems to them
  - $\displaystyle \frac{d}{dt}\iiint_V \rho \ dV = \iiint_V \frac{\partial\rho}{\partial t} \ dV$
  - divergence theorem

$$\iint_{\partial V}\vec{\phi}\cdot\vec{n} \ dA = \iiint_V \vec{\nabla}\cdot\vec{\phi} \ dV$$

  Then global conservation law becomes

$$\iiint_V \left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{\phi} - f\right) dV = 0$$

- V is arbitrary
- if integrand is continuous
  local conservation law: $\displaystyle \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{\phi} = f$ $\quad\rightarrow\quad$ works for all transport

special case if ρ depends only on x, t and $\vec{\phi}$ goes in $\vec{x}$ direction

$$\frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} = f$$