AMATH 353 final review

PDE Comprehensive Notes

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Preface

This review notes consist of two parts: ERV (Winter 2010) and Dr. Giuseppe Sellaroli (Spring 2019). They focused on different subjects, but main idea is same. For the Spring 2019 part, the content will be more comprehensive. The content might vary from term to term. Use at your own risk.

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Contents

Winter 2010

1. Models

1.1 heat eqn

- Conduction: (传导) the transfer of vibrational energy from one molecule or atom to another by means of collisions. The molecules/atoms do not move appreciably. This is the primary mechanism for heat flow in solids.
- Convection: (对流) the molecules/atoms move appreciably from one place to another, taking the thermal energies with them. This will occur in gases and liquids.

1.1.1 Thermal density

- $e(x, t)$: thermal energy density
- Δh: amount of heat
- A: area

Ideally, we define

$$e(x,t) = \lim_{\Delta x \to 0} \frac{\Delta h}{A \Delta x}$$

Then

$$h(a, b; t) = \int_a^b e(x,t) A \, dx$$

1.1.2 Conservation

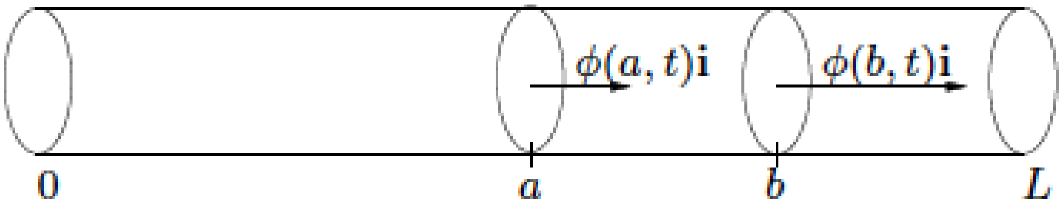
rate of change of total thermal energy
=
net heat flow across boundaries per unit time
+
total thermal energy generated inside per unit time

$$LHS = \frac{dh(a,b;t)}{dt} = \frac{d}{dt} \int_a^b e(x,t) A dx = \int_a^b \frac{\partial e(x,t)}{\partial t} A dx$$

1.1.3 Heat Flux

$\phi(x,t)$: this is the amount of thermal energy per unit time flowing to the right per unit surface area.

- if > 0, then heat is flowing to the right at x.
- < 0, then ... left ...



1.1.4 Internal heat sources

$Q(x,t)$: thermal energy per unit volume generated per unit time at point x and time t.

$$\int_a^b \frac{\partial e(x,t)}{\partial t} A dx = \phi(a,t)A - \phi(b,t)A + \int_a^b Q(x,t) A dx$$

$$\int_a^b \left[\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q\right] dx = 0$$

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0$$

Temperature function $u(x,t)$.

Specific heat, c: the heat energy that must be applied to a unit mass of a substance to raise its temperature by one unit

Then heat energy per unit mass is given by

$$c(x)[u(x,t) - u_0]$$

mass

$$\Delta m = \rho \Delta V = \rho A \Delta x$$

total energy

$$e(x,t) A \Delta x = c(x)\left[u(x,t) - u_0\right] \rho(x) A \Delta x$$

Divide by $A\Delta x$ and substitute to the equation above, we have

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

Theorem 1.1.1 — Fourier's law of heat conduction.

$$\phi = -K_0 \frac{\partial u}{\partial x}$$

where K_0 is known as the thermal conductivity substitute the result for ϕ into previous equation

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q$$

If $Q = 0$, then PDE becomes

Theorem 1.1.2 — Heat Equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where $k = \frac{K_0}{c\rho}$ is known as thermal diffusivity.

The conservation of mass principle states that the rate of change of chemical in $[a, b]$ will be equal to the net flow of chemical into the interval

$$\frac{d}{dt}\int_a^b u(x,t)Adx = [\phi(a,t) - \phi(b,t)]A$$

then we may conclude that

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}, \quad 0 \leq x \leq L$$

Theorem 1.1.3 — Fick's law of diffusion.

$$\phi(x,t) = -k\frac{\partial u}{\partial x}$$

where $k > 0$ is the coefficient of diffusivity.

If we substitute, then diffusion equation yields to the same form.

1.1.5 higher dimension

$e(\vec{x}, t)$: the thermal energy density: (heat per unit volume) at a point $\vec{x} \in V$ in the solid.
$\phi(\vec{x}, t)$: the heat flux vector at a point $\vec{x} \in V$.
Then the total heat energy:

$$\iiint_D e(\vec{x}, t)dV$$

basic idea:

$$\begin{array}{c}\text{rate of change}\\\text{of total heat}\\\text{energy in time}\end{array} = \begin{array}{c}\text{net heat flow}\\\text{across boundary}\\\text{per unit time}\end{array} + \begin{array}{c}\text{heat energy}\\\text{generated inside}\\\text{per unit time}\end{array}$$

$$LHS = \frac{d}{dt}\iiint_D e(\mathrm{x}, t)dV = \iiint_D \frac{\partial e}{\partial t}dV$$

$$RHS = -\iint_S \Phi \cdot \hat{\mathbf{n}}dS + \iiint_D Q\ dV$$

By Divergence Theorem:

$$\iint_S \Phi \cdot \hat{\mathbf{n}}dS = \iiint_D \vec{\nabla} \cdot \Phi dV$$

Then combine all terms, we have the conservation equation

$$\frac{\partial e}{\partial t} = -\vec{\nabla} \cdot \Phi + Q$$

$Q = 0$, together with

Theorem 1.1.4 — Fick's law of transport.

$$\phi = -k\vec{\nabla} \cdot u$$

we have the three dimensional diffusion equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

1.1.6 IC and BC

1. Initial condition: $u(x, 0) = g(x), \quad 0 \leq x \leq L$.
2. Boundary conditions:

$$u(0, t) = T_1, \quad u(L, t) = T_2$$

   or time-varying temp:

$$u(0, t) = f_B(t), \quad u(L, t) = g_B(t)$$

1.1.7 equilibrium

$u = u_{eq}(x) \implies \frac{\partial u}{\partial t} = 0$

2. Methods of Separation of Variables

2.1 Heat Equation

Recall 1D heat equation without sources

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with zero-endpoint BCs

$$u(0,t) = 0, \quad u(L,t) = 0$$

and IC

$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

We assume a solution of the form

$$u(x,t) = \phi(x)G(t)$$

We introduce a separating constant $\mu \in \mathbb{R}$:

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

Then we discuss in three cases $\lambda = 0, > 0, < 0$.

2.2 Laplace

Recall Laplace's equation for a function $u : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\Delta^2 u = 0$$

with 4 BCs

2.2.1 Rectangular Region

Just as usual

2.2.2 Circular

It is more simple to work in planar polar coordinates (r, θ) so that $u = u(r, \theta)$. Then Laplace's equation becomes

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Because of this singularity at $t = 0$, we'll also need a condition on solutions there: With an eye to physical applications, we impose the condition of boundedness

$$|u(0, \theta)| < \infty$$

2.3 Sturm-Liouville Theory

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a < x < b$$

subject to general homogeneous boundaries of the form

$$\beta_1\phi(a) + \beta_2\frac{\phi}{dx}(a) = 0$$
$$\beta_3\phi(b) + \beta_4\frac{\phi}{dx}(b) = 0$$

several conditions

1. $p(x)$ piecewise C^1, $q(x)$ and $\sigma(x)$ are piecewise continuous
2. $p(x) > 0$, $\sigma(x) > 0$

3. Fourier Transform Solutions of PDEs

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega$$

The Fourier transform of a Gaussian $f(x) = e^{-ax^2}$ is a Gaussian $F(\omega) = \frac{1}{\sqrt{4\pi a}}e^{-\frac{1}{4a}\omega^2}$

3.1 Solution of Heat Equation via Fourier Transforms and Convolution Theorem

The complete notes can be found here

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty$$

with IC

$$u(x,0) = f(x)$$

has solution

$$u(x,t) = \int_{-\infty}^{\infty} f(s)h_t(x-s)ds, \quad t > 0$$

where the “heat kernel" function $h_t(x)$ is given by

$$h_t(x) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}, \quad t > 0$$

3.2 Parseval's Identity

skipped

3.3 Dirac delta function

Recall that delta function $\delta(x)$ is not a function in the usual sense. It has following properties

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

with the additional feature that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

It's an example of a distribution: distributions are defined in terms of their integration properties.

4. Quasilinear PDEs

A quasilinear PDEs has the form

$$\frac{\partial u}{\partial t} + c(u,x,t)\frac{\partial u}{\partial x} = Q(u,x,t)$$

It is called quasilinear because the partial derivatives do not multiply each other. Note that the PDE can be nonlinear since the coefficient c could be a function of u. As well, the function Q could be nonlinear in u. The special form of the quasilinear PDE permits its reduction to a system of ODEs which can, at least in principle, be solved, as we show here.

4.1 Application

traffic flow

4.2 Shock Waves

here.

II

Spring 2019

5. Basics & Models

5.1 Basics

Definition 5.1.1 A PDE is an equation that relates an unknown function of two or more variables to its partial derivatives.

Definition 5.1.2 — order. The order of a PDE is defined to be the order of the highest order derivative appearing in it.

■ **Example 5.1**

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$$

is first order, while

$$\frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^3 = 3$$

is second-order. ■

Definition 5.1.3 — linear. A PDE is linear if the unknown u and its partial derivatives appear alone and to the first power; their coefficients are allowed to depend on the independent variables (x, y, etc)

■ **Example 5.2**

$$u + \frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 3x$$

is linear, while

$$u^2 + \sin\left(\frac{\partial u}{\partial x}\right) = 0$$

is non-linear. ■

Definition 5.1.4 — homogeneous. A <u>linear</u> PDE is called homogeneous if every term contains the unknown u or one of its derivatives.

■ **Example 5.3**

$$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + u = 0$$

is homogeneous, while

$$\frac{\partial u}{\partial x} + y = 0$$

is non-homogeneous. Note that

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$$

is not homogeneous (despite its appearence) since it is not linear. ■

5.2 Models

5.2.1 1D conservation law

Idea is same as before. We have

- quantity $Q(t)$
- density $\rho(x,t)$
- source term $f(x,t,\rho)$
- flux $\phi(x,t)$

Then we have local conservation law

$$\frac{\partial \rho}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) = f(x,t,\rho)$$

5.2.2 3D conservation law

Global

$$Q(t) = \iiint_V \rho(\vec{r},t)\,\mathrm{d}V$$

The net rate of change of Q due to transport, which we denote by $T(t)$, is given by the net rate at which quantity enters V through the boundary ∂V. To describe $T(t)$ locally we introduce the flux $\vec{\phi}(\vec{r},t)$, which this time we make into a vector field. Then we can write $T(t)$ as the surface integral

$$T(t) = -\iint_{\partial V} \vec{\phi}(\vec{r},t)\cdot\vec{n}(\vec{r})\,\mathrm{d}A$$

The net rate of change of Q due to sources, which we denote by $S(t)$, can be written as

$$S(t) = \iiint_V f(\vec{r},t,\rho(\vec{r},t))\,\mathrm{d}V$$

Then the global conservation law for this process is given by

$$\frac{dQ}{dt} = T(t) + S(t)$$

Local

or, in terms of the integrals, we have

$$\frac{d}{dt}\iiint_V \rho(\vec{r},t)\,\mathrm{d}V = -\iint_{\partial V} \vec{\phi}(\vec{r},t)\cdot\vec{n}(\vec{r})\,\mathrm{d}A + \iiint_V f(\vec{r},t,\rho(\vec{r},t))\,\mathrm{d}V$$

Then by divergence theorem, finally, we have local conservation law

$$\frac{\partial \rho}{\partial t}(\vec{r},t) + \nabla\cdot\vec{\phi}(\vec{r},t) = f(\vec{r},t,\rho(\vec{r},t))$$

Problem 5.1 What is the difference between global and local conservation laws?

Answer

Global conservation laws describes the change in the quantity of a physical system, and so the equation used to describe it an ODE evaluating only one variable that describes the entire system. Local conservation laws on the other hand take different variables that construct the system into account and so the equation consists of PDEs describing the change in those variables. For example, In the lecture we are given an example of heat. We can only consider time as the variable for total system, but if we zoom in for space in order to get more information, we need to apply divergence th and then we can get the information about space, i.e. r variable here. And that gives us a pde.

Considering then initial conditions, we have linear combination of separable solutions. To let the solution make sense, we must have the complete set, otherwise we will have infinitely many solutions.

5.3 Diffusion equation

we have Fick's law

$$\vec{\phi} = -D\nabla\rho,$$

Note that, despite its name, Fick's law is a model approximating reality, not a rule set in stone. Putting together Fick's law and the conservation law, we get the diffusion equation

$$\frac{\partial\rho}{\partial t} = D\nabla^2\rho$$

where $D > 0$ is called the diffusion constant.

5.4 Heat equation

we know[1] from thermodynamics that

$$u = \rho CT$$

where u is density of thermal energy, rho is mass density, C is the specific heat of the object, and T is the temperature at each point and times.

we have Fourier's law

$$\vec{\phi} = -K\nabla T$$

where $K > 0$ is called thermal conductivity. [2] Putting these together we have heat equation

$$\frac{\partial T}{\partial t} = k\nabla^2 T$$

where $k = K/\rho C$ is the thermal diffusivity.

5.5 Advection equation

We call advection (对流) the transport of a quantity due to the motion of a fluid that carries it. Note that this is different than diffusion, where for example a chemical moves in a static fluid.

[1] I don't :)

[2] We are going to assume that K is constant for simplicity, but it is an approximation: in reality it depends on the temperature.

Given the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\phi} = 0$$

for the quantity being transported, we can model advection by using constitutive relation

$$\vec{\phi} = \rho \vec{v}$$

hat is, we say that the flux is directly proportional to both the density of the substance and the speed of the fluid, and it follows the flow of motion of the fluid. Thus we get the PDE

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

If we further assume that the fluid is incompressible, a result from fluid mechanics (we are not going to prove this in class, but feel free to ask me about the proof at office hours or during tutorial) tells us that $\nabla \cdot \vec{u} = 0$, from which it follows that

$$\nabla \cdot (\rho \vec{v}) = \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} = \vec{v} \cdot \nabla \rho$$

so we have advection equation

Definition 5.5.1 — advection equation.

$$\frac{\partial \rho}{\partial t}(\vec{r}, t) + \vec{v}(\vec{r}, t) \cdot \nabla \rho(\vec{r}, t) = 0$$

In the special case in which ρ depends only on x, t and the velocity field is constant and the velocity field is constant and directly along the x-axis, i.e.

$$\vec{v} = c\,\hat{x}, \quad c \in \mathbb{R},$$

the advection equation reduces to

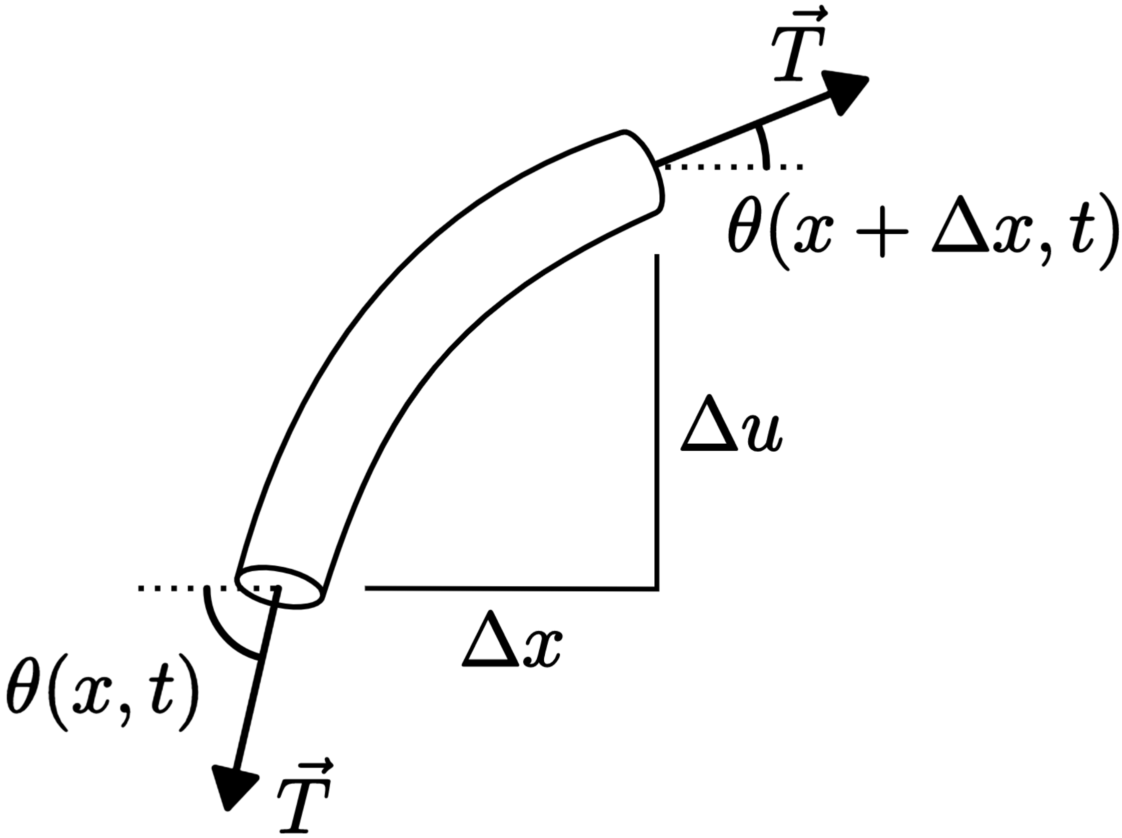
$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0.$$

5.5.1 Nomenclature

- terms proportional to $\nabla^2 \rho$ are called diffusion terms
- terms proportional to $\nabla \rho$ are called advection terms
- source terms are also called reaction terms, as they often represent creation or destruction by means of chemical reactions

5.6 Wave equations

5.6.1 String Vibrations



The vertical component of Newton's second law is approximately

$$m\frac{\partial^2 u}{\partial t^2} \approx -T\sin(\theta(x,t)) + T\sin(\theta(x+\Delta x,t))$$

mass is approximately

$$m \approx \rho A\sqrt{(\Delta x)^2 + (\Delta u)^2} = \rho A\Delta x\sqrt{1+\left(\frac{\Delta u}{\Delta x}\right)^2}$$

Then by some approximations... we have

Definition 5.6.1 — wave equation.

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

where

$$c = \sqrt{\frac{T}{\rho A}}$$

has units of speed

5.6.2 Vibrating membrane

we assume:

- membrane density ρ is constant
- thickness h is constant
- the tension per unit length $\vec{\tau}$ of the membrane which is a vector field defined at each point of the membrane. We assume it has constant magnitude τ.

By Stoke's Theorem and some approximation we have

Definition 5.6.2 — wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

where

$$c = \sqrt{\frac{\tau}{\rho h}}$$

has units of speed.

6. Classification of 2nd order PDEs

6.1 Generic 2nd order linear PDE in two variables

Definition 6.1.1 — generic form.

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

Note that we assume u is of class $\mathcal{C}^2$.

Definition 6.1.2 — Discriminant.

$$\operatorname{Disc}(x, y) = B^2 - AC$$

- if Disc > 0, hyperbolic
- if = 0, parabolic
- if < 0, elliptic

6.2 Canonical form of the wave equation and its solutions

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = 0$$

assuming u is of class $\mathcal{C}^2$ so that the mixed derivatives cancel out.

By changing of variables

$$\begin{cases} x = \frac{1}{2}(\eta + \xi) \\ t = \frac{1}{2c}(\eta - \xi) \end{cases}$$

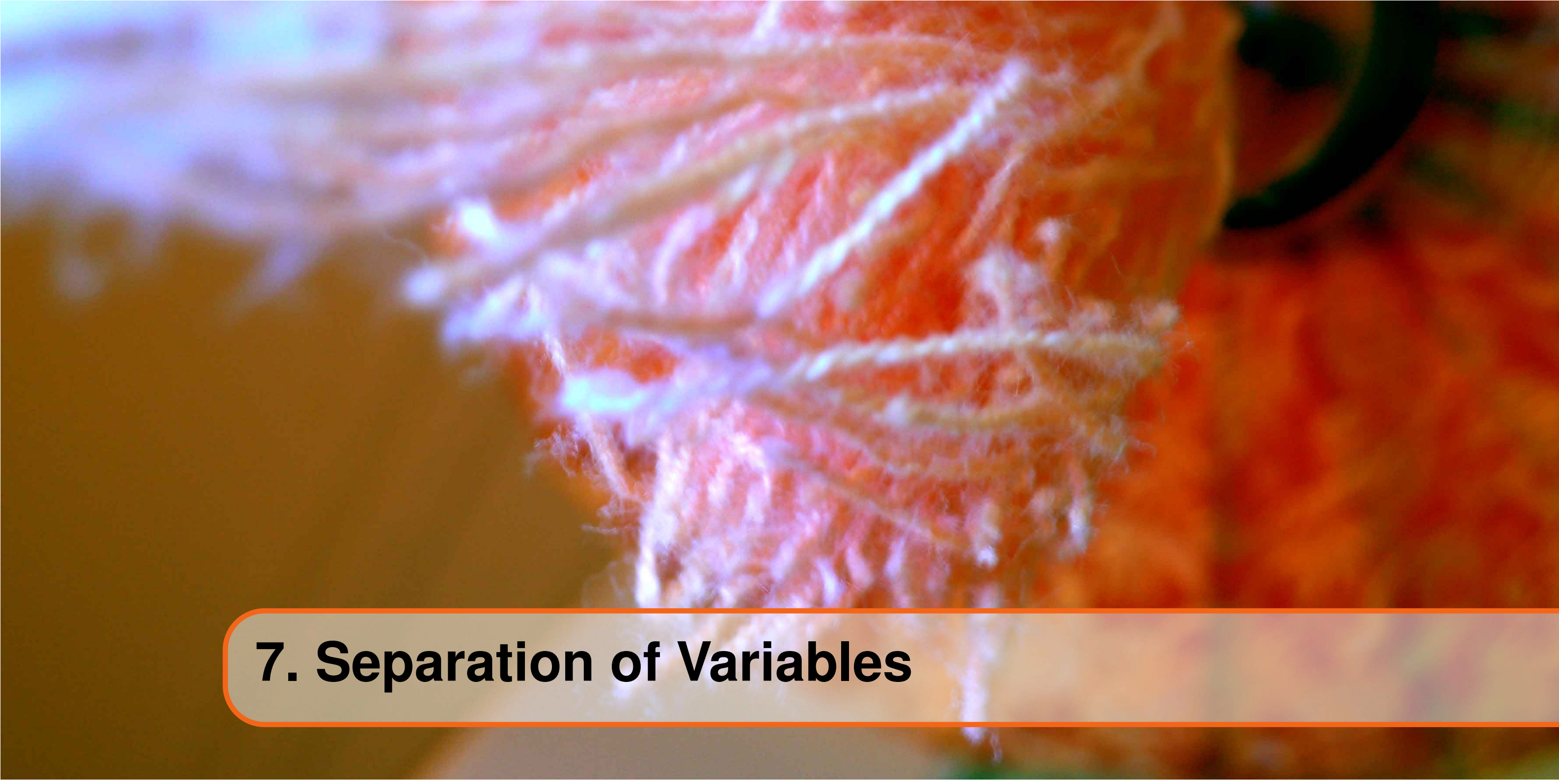
Then we have

$$\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0$$

Therefore the general solution is

$$u(x,t) = \alpha(x - ct) + \beta(x + ct)$$

We say $\alpha(x - ct)$ is a right-travelling wave, since if we plot it as a function of x and let t increase, the plot shift to the right at constant speed c (assuming $c > 0$). Likewise $\beta(x + ct)$ os a left-travelling wave.



7. Separation of Variables

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0 \quad \forall t \geq 0$$

$$\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x), \end{cases} \qquad x \in [0,L], \tag{7.1}$$

Separate:

$$u(x,t) = M(x)N(t)$$

$$\frac{1}{c^2}N''(t)M(x) = M''(x)N(t)$$

$$M(0) = M(L) = 0$$

we are going to assume that $M(x)$ and $N(t)$ are both not always zero. Under this assumption, we can write general solution as the following system:

$$\begin{cases} M''(x) = -\lambda M(x) \\ N''(t) = -c^2\lambda N(t) \end{cases}$$

Then we discuss $\lambda = 0, -s^2, s^2$ with $s > 0$. Then we have our general solution

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L}\right)\left[A_k \cos\left(\frac{ck\pi t}{L}\right) + B_k \sin\left(\frac{ck\pi t}{L}\right)\right]$$

We need the function to satisfy the initial conditions (7.1) if and only if

$$f(x) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{L}\right) \tag{7.2}$$

$$g(x) = \sum_{k=1}^{\infty} \frac{ck\pi}{L} B_k \sin\left(\frac{k\pi x}{L}\right). \tag{7.3}$$

Then we can always find unique coeff A_k and B_k by fourier sine series:

$$A_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x \tag{7.4}$$

$$B_k = \frac{2}{ck\pi}\int_0^L g(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x. \tag{7.5}$$

7.1 IC & BC

7.1.1 Boundary Conditions

1. Dirichlet conditions specify the value of the unknown u on the boundary. $u(0,t) = 0$ $u(L,t) = \sin(3t)$
2. Neumann conditions specify the value of the spatial derivative of u on the boundary/
3. Robin conditions specify the value of a combination of both u and the spatial derivative of u on the boundary

7.1.2 Initial conditions

Whenever the unknown function u depends on time, we also need to impose initial conditions to make sure that we have enough data to find a unique solution.

7.1.3 Well-posed problems

Following the definition given by Hadamard, we say that a BVP/IVP/IBVP is well posed if:

1. the problem has a solution;
2. the solution is unique;
3. the behaviour of the solution depends continuously on the initial/boundary conditions—in other words, if we change the data a little bit we don't want a drastically different solution;

The first requirement is self-explanatory: a model without solutions is not very useful. The second and third requirements are more subtle: we want unique solutions that depend continuously on the data since we often need to solve PDEs numerically, and we need to make sure of two things:

- there is only one solution to pick from, so that numerical errors don't send us to the wrong solution;
- numerical errors and approximations in the data don't radically change the solution from the real one.

7.2 Sturm–Liouville theory

Definition 7.2.1 A function $f : [a,b] \to \mathbb{R}$ is square-integrable on $[a,b]$ with weight $w(x)$ if

$$\int_a^b |f(x)|^2 w(x)\,\mathrm{d}x < \infty, \tag{7.6}$$

where $w(x) > 0$ on $[a,b]$.

Definition 7.2.2 An infinite sequence of functions $\{\varphi_n\}_{n=1}^{\infty} \subset L^2([a,b],w)$ is called a complete orthonormal set if

$$\langle \varphi_m, \varphi_n \rangle = \delta_{mn} \quad \text{(orthonomality)} \tag{7.7}$$

and for any $f \in L^2([a,b],w)$ we have

$$f = \sum_{n=1}^{\infty} \langle \varphi_n, f \rangle \, \varphi_n \quad \text{(completeness)}. \tag{7.8}$$

The r.h.s. of (7.8) is called the generalised Fourier series of f, and the equality is to be understood as "the generalised Fourier series of f converges to f".

Definition 7.2.3 — Sturm–Liouville problem. Let L be the operator on $L^2([a,b],w)$ defined by

$$(Lf)(x) = \frac{1}{w(x)} \left[p(x) f''(x) + p'(x) f'(x) + q(x) f(x) \right] \tag{7.9}$$

for some fixed functions $p(x) > 0$ and $q(x)$, with p of class $\mathcal{C}^1$ (continuously differentiable) and q continuous. More compactly, we can write

$$L = \frac{1}{w(x)} \left[p(x) \frac{d^2}{dx^2} + p'(x) \frac{d}{dx} + q(x) \right]. \tag{7.10}$$

A (regular) Sturm–Liouville problem consists in finding the eigenfunctions and eigenvalues of L,

$$Lf = -\lambda f \quad \text{(the minus sign is by convention)} \tag{7.11}$$

subject to the constraints

$$\alpha_1 f(a) + \alpha_2 f'(a) = 0 \qquad \alpha_1^2 + \alpha_2^2 \neq 0 \tag{7.12}$$

$$\beta_1 f(b) + \beta_2 f'(b) = 0 \qquad \beta_1^2 + \beta_2^2 \neq 0. \tag{7.13}$$

In other words, we are trying to find out for which values of λ the ordinary differential equation

$$p(x) f''(x) + p'(x) f'(x) + q(x) f(x) = -\lambda w(x) f(x) \tag{7.14}$$

has a nonzero solution satisfying the constraints (7.12) and (7.13).

There are some properties:

1. eigenvalues are all real, bounded below but not above
2. The eigenspaces are 1-dimensional, i.e., each eigenvalue λ_n has a unique (up to a multiplicative factor) eigenfunction φ_n. Moreover the function φ_n has exactly $n-1$ zeroes in the open interval (a,b).
3. The eigenfunctions form a complete orthonormal set, assuming they have been normalised to ensure that

$$\|\varphi_n\| = 1. \tag{7.15}$$

Theorem 7.2.1 — non-negative eigenvalues. Consider the Sturm–Liouville problem

$$\begin{cases} Lf = -\lambda f \\ \alpha_1 f(a) + \alpha_2 f'(a) = 0 \\ \beta_1 f(b) + \beta_2 f'(b) = 0 \end{cases} \tag{7.16}$$

with

$$L = \frac{1}{w(x)}\left[p(x)\frac{d^2}{dx^2} + p'(x)\frac{d}{dx} + q(x)\right], \quad p(x), w(x) > 0, \quad x \in [a,b]. \tag{7.17}$$

If we have

$$\begin{cases} q(x) \leq 0 \\ \alpha_1\alpha_2 \leq 0 & (\alpha_1 \text{ and } \alpha_2 \text{ have opposite sign}) \\ \beta_1\beta_2 \geq 0 & (\beta_1 \text{ and } \beta_2 \text{ have the same sign}) \end{cases} \tag{7.18}$$

then λ must be non-negative, i.e., $\lambda \geq 0$.

Proof. omitted ■

7.2.1 Counterexample: what can happen when Sturm–Liouville theory doesn't apply

$$u(0,t) + u(\pi,t) = 0 \quad t \geq 0$$

the boundary conditions are not of the Sturm–Liouville kind, since one of them mixes a condition at $x = 0$ with one at $x = \pi$.

7.2.2 When and why is separation of variables justified as a method to find the most general solution of a PDE?

Sturm–Liouville theory

If the PDE and the boundary conditions are of the Sturm–Liouville form, we are guaranteed to find the most general solution using the method of separation of variables.

(i.e. whether the normal modes form a COMPLETE orthonormal set when t = 0 to satisfy any possible initial condition or not, or in other words, can every possible solution of the PDE be written as a linear combination of the normal modes) One of the properties of Sturm-Liouville Problems is that its eigenfunctions form a complete orthonormal set, so given any reasonably nice initial condition, it can be written as a linear combination of the eigenfunctions and thus the normal modes when t = 0. Otherwise, there will be certain initial condition that cannot be wriiten by a linear combination of the separable solutions.

It is justified in the case when the sturm-liouville theory can be applied given the form of equation and boundary conditions. The theory guarantees the solution in the form of separable functions. t is justified in the case when the sturm-liouville theory can be applied given the form of equation and boundary conditions. The theory guarantees the solution in the form of separable functions. The final form of the equation gets into that of the ratios which is equal to a separation constant; that Tcan be separated into two eequations. SL theory guarantees the solution of equations in that form.

7.3 More examples of S o V

Vibrating circular membrane

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad x^2 + y^2 < R^2, \quad t > 0 \tag{7.19}$$

and the boundary conditions

$$u(x, y, t) = 0 \qquad x^2 + y^2 = R^2 \tag{7.20}$$

Because of the symmetry, we can use polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 < r \le R, \quad 0 \le \theta \le 2\pi. \tag{7.21}$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 < r \le R, \quad 0 \le \theta \le 2\pi. \tag{7.22}$$

In these coordinates the PDE becomes

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \tag{7.23}$$

while the boundary conditions become

$$u(R, \theta, t) = 0 \qquad \theta \in [0, 2\pi], \quad t \ge 0. \tag{7.24}$$

Note that when $\lambda > 0$, we will see the Bessel equation. Taking a linear combination of all the normal modes, we get the candidate general solution

$$u(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\left[A_{nk}\cos(n\theta) + B_{nk}\sin(n\theta)\right] J_n\left(\frac{j_{nk} r}{R}\right)\sin(\omega_{nk} t + \varphi_{nk}), \tag{7.25}$$

8. Inhomogeneous

8.1 Duhamel's principle

Robin Boundary conditions:

$$\mathcal{B}u = \begin{bmatrix} \alpha_1 u(a,t) + \alpha_2 \frac{\partial u}{\partial x}(a,t) \\ \beta_1 u(b,t) + \beta_2 \frac{\partial u}{\partial x}(b,t) \end{bmatrix} \tag{8.1}$$

Suppose we want to solve the IBVP

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + F(x,t) & a < x < b, \quad t > 0 \\ \mathcal{B}u = \vec{0} & t \geq 0 \\ u(x,0) = 0 & a \leq x \leq b \end{cases} \tag{8.2}$$

where L is differential operator only has derivatives w.r.t. x.

Theorem 8.1.1 — Duhamel's principle, 1st order. The solution to the IBVP (8.2) is given by

$$u(x,t) = \int_0^t v_s(x, t-s)\,\mathrm{d}s, \tag{8.3}$$

where $\{v_s\}_{s\geq 0}$ is a family of functions satisfying the IBVPs

$$\begin{cases} \dfrac{\partial v_s}{\partial t} = Lv_s & a < x < b, \quad t > 0 \\ \mathcal{B}v_s = \vec{0} & t \geq 0 \\ v_s(x,0) = F(x,s) & a \leq x \leq b. \end{cases} \tag{8.4}$$

Note that the principle only works if v_s depends continuously on the parameter s.

Proof. omitted ■

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \alpha(t)\dfrac{\partial u}{\partial t} = Lu + F(x,t) & a < x < b, \quad t > 0 \\ \mathcal{B}u = \vec{0} & t \geq 0 \\ u(x,0) = 0 & a \leq x \leq b \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & a \leq x \leq b \end{cases} \tag{8.5}$$

Theorem 8.1.2 — Duhamel's principle, 2nd order. The solution to the IBVP (8.5) is given by

$$u(x,t) = \int_0^t v_s(x, t-s)\,\mathrm{d}s, \tag{8.6}$$

where each function v_s satisfies

$$\begin{cases} \dfrac{\partial^2 v_s}{\partial t^2} + \alpha(t)\dfrac{\partial v_s}{\partial t} = Lv_s & a < x < b, \quad t > 0 \\ \mathcal{B}v_s = \vec{0} & t \geq 0 \\ v_s(x,0) = 0 & a \leq x \leq b \\ \dfrac{\partial v_s}{\partial t}(x,0) = F(x,s) & a \leq x \leq b. \end{cases} \tag{8.7}$$

Note that the principle only works if v_s depends continuously on the parameter s.

Proof. omitted ■

Definition 8.1.1 — resonance. This phenomenon is known as resonance: if the string is subject to a forcing term which oscillates at a frequency close to the frequency of one of the normal modes, there is a positive feedback effect and the string can vibrate considerably. This is how, for example, a glass can be shattered by making it vibrate at the right frequency.

8.2 Homogenize non-zero initial conditions

8.2.1 first order

Suppose that we want to solve the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + F(x,t) & a < x < b, \quad t > 0 \\ \mathcal{B}u = \vec{0} & t \geq 0 \\ u(x,0) = f(x) & a \leq x \leq b, \end{cases} \tag{8.8}$$

where L is a differential operator involving x only and $\mathcal{B}$ is the generic boundary operator we defined a few lectures ago.

We don't know how to solve this directly, so we are going to introduce a new function $\psi(x,t)$ which satisfies

$$\psi(x,0) = 0 \tag{8.9}$$

and which is related to $u(x,t)$ in a simple way.

The simplest choice is

$$\psi(x,t) = u(x,t) - f(x). \tag{8.10}$$

8.2.2 second order

Suppose that we want to solve the equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \alpha(t)\dfrac{\partial u}{\partial t} = Lu + F(x,t) & a < x < b, \quad t > 0 \\ \mathcal{B}u = \vec{0} & t \geq 0 \\ u(x,0) = f(x) & a \leq x \leq b \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & a \leq x \leq b. \end{cases} \tag{8.11}$$

Just as before, we want to introduce a new function $\psi(x,t)$ which satisfies

$$\psi(x,0) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial t}(x,0) = 0 \tag{8.12}$$

and is related to $u(x,t)$ in a simple way.

The simplest choice is

$$\psi(x,t) = u(x,t) - f(x) - g(x)t \tag{8.13}$$

8.3 Homogenize non-zero boundary conditions

Suppose we want to solve some PDE with unknown $u(x,t)$, with boundary conditions

$$\begin{cases} \alpha_1 u(a,t) + \alpha_2 \dfrac{\partial u}{\partial x}(a,t) = A(t) \\ \beta_1 u(b,t) + \beta_2 \dfrac{\partial u}{\partial x}(b,t) = B(t). \end{cases} \tag{8.14}$$

In order to transform the problem into one we know how to solve, we introduce the new unknown

$$\psi(x,t) = u(x,t) - u_0(x,t), \tag{8.15}$$

where u_0 is any function that satisfies the boundary conditions (8.14), i.e.,

$$\begin{cases} \alpha_1 u_0(a,t) + \alpha_2 \dfrac{\partial u_0}{\partial x}(a,t) = A(t) \\ \beta_1 u_0(b,t) + \beta_2 \dfrac{\partial u_0}{\partial x}(b,t) = B(t). \end{cases} \tag{8.16}$$

Then we can find (and solve) the PDE for ψ, which has homogeneous boundary conditions by construction. The process of replacing u with ψ is called "homogenisation of the boundary conditions".

Note that the function u_0 only needs to satisfy the boundary conditions: it has no PDE or initial conditions. There are usually (infinitely) many choices for u_0, but the goal is to find a simple one.

We usually let $u_0(x,t) = \alpha(t) + x\beta(t) + x^2\gamma(t)$

8.4 Eigenfunction expansion

Duhamel's principle is a very useful method, but it can't be used with time-independent PDEs (such as Laplace's equation). To overcome this problem, we will introduce a different method to solve inhomogeneous PDEs on bounded domains, known as "eigenfunction expansion" or "finite Fourier transform". It is essentially a generalisation of separation of variables, that also works for nonhomogeneous equations.

Steps

- Find a complete orthogonal set $\{M_k(x)\}_{k=1}^{\infty}$ satisfying

$$\begin{cases} LM_k = -\lambda_k M_k \\ \mathcal{B}M_k = \vec{0} \end{cases}$$

- The trick is to write also functions that depend on both x and t as linear combinations of the M_k (that's the eigenfuction expansion).
- Then solve the IBVP termwisely.

8.5 Duhamel's principle vs eigenfunction expansion

So what's the difference between the two methods we have seen? Not much really, it's more of a change in the order in which we do things:

- when we use duhamel's principle, we first deal with the fact that the equation is non-homogeneous, then use separation of variables (a subcase of eigenfunction expansion) to reduce the problem to a system of ODEs;
- when we use eigenfunction expansion, we first reduce the problem to a system of non-homogeneous ODEs and then we deal with the inhomogeneity (often using duhamel's principle!).

Eigenfunction expansion has advantages in some cases though:

- it can be used even if there is no time variable (technically speaking Duhamel's principle can be used as well, but it's not as straightforward as usual);
- we can use a variety of techniques to solve the inhomogeneous ODEs, we don't have to necessarily use Duhamel's principle.

9. Unbounded domains & Fourier transform

Lebesgue integration, Dirac's delta

9.1 Fourier Transform

$$\mathcal{F}[f](k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-ikx}\, \mathrm{d}x \tag{9.1}$$

$$\mathcal{F}^{-1}[f](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(k)\, e^{ikx}\, \mathrm{d}k \tag{9.2}$$

shorthand notations:

$$\hat{f} = \mathcal{F}[f], \quad \check{f} = \mathcal{F}^{-1}[f]. \tag{9.3}$$

9.2 Convolution Theorem

$$(f * g)(x) = \int_{\mathbb{R}} f(x-s)g(s)\, \mathrm{d}s. \tag{9.4}$$

Theorem 9.2.1 — convolution theorem.

$$\mathcal{F}[f * g](k) = \sqrt{2\pi}\hat{f}(k) \cdot \hat{g}(k) \tag{9.5}$$

$$\mathcal{F}^{-1}[f * g](k) = \sqrt{2\pi}\check{f}(k) \cdot \check{g}(k). \tag{9.6}$$

The convolution theorem also works the other way around, that is

$$\mathcal{F}[f\cdot g](k) = \frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(k) \tag{9.7}$$

$$\mathcal{F}^{-1}[f\cdot g](k) = \frac{1}{\sqrt{2\pi}}(\check{f} * \check{g})(k), \tag{9.8}$$

Theorem 9.2.2 — Fourier transform of derivatives.

$$\mathcal{F}\left[\frac{d^n f}{dx^n}\right](k) = (ik)^n \hat{f}(k). \tag{9.9}$$

T, distribution. Recall that distributions are (linear) functions acting on other functions.

■ **Example 9.1 — heat equation.**

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} & x\in\mathbb{R}, \quad t>0 \\ u(x,0) = f(x) & x\in\mathbb{R} \end{cases} \tag{9.10}$$

where $f(x)$ is the initial temperature at $t=0$ and $D>0$.

The idea is to take the Fourier transform with respect to x of all function involved in the IVP.

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(x,t)\, e^{-ikx}\,\mathrm{d}x \tag{9.11}$$

$$\widehat{\frac{\partial u}{\partial t}}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\partial u}{\partial t}(x,t)\, e^{-ikx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{\mathbb{R}} u(x,t)\, e^{-ikx}\,\mathrm{d}x = \frac{\partial \hat{u}}{\partial t}(k,t) \tag{9.12}$$

$$\widehat{\frac{\partial^2 u}{\partial x^2}}(k,t) = -k^2\hat{u}(k,t) \tag{9.13}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\, e^{-ikx}\,\mathrm{d}x \tag{9.14}$$

since Fourier Transform is linear, the BVP becomes

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} = -Dk^2\hat{u} \\ \hat{u}(k,0) = \hat{f}(k) \end{cases} \tag{9.15}$$

■

10. Method of characteristics

The idea behind the method of characteristics
a surface S in the xtz-space through equation $z = u(x,t)$

10.1 general approach

Consider the Cauchy problem

$$\begin{cases} a(x,t,u)\dfrac{\partial u}{\partial x} + b(x,t,u)\dfrac{\partial u}{\partial t} = c(x,t,u) \\ u(\gamma_1(r), \gamma_2(r)) = f(r). \end{cases} \tag{10.1}$$

Just as before, we want to rewrite the surface

$$z = u(x,t) \tag{10.2}$$

as a parametric surface

$$\begin{cases} x = x(r,s) \\ t = t(r,s) \\ z = z(r,s) \end{cases} \tag{10.3}$$

built up from characteristic curves. This time the characteristic equations are

$$\begin{cases} \dfrac{\partial x}{\partial s} = a(x,t,z) \\ \dfrac{\partial t}{\partial s} = b(x,t,z) \\ \dfrac{\partial z}{\partial s} = c(x,t,z) \end{cases} \qquad \begin{cases} x(r,0) = \gamma_1(r) \\ t(r,0) = \gamma_2(r) \\ z(r,0) = f(r). \end{cases} \tag{10.4}$$

‘ Once the parametric surface has been found, we try to find a function F such that

$$(r,s) = F(x,t), \tag{10.5}$$

at least in a neighbourhood of the initial curve Γ. Once this is done, we can write the solution as

$$u(x,t) = z(F(x,t)). \tag{10.6}$$

■ Example 10.1 — inviscid Burger's equation.

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases} \tag{10.7}$$

which is an non-linear advection equation in which the fluid velocity is proportional to $u(x,t)$.

The characteristic equations are

$$\begin{cases} \dot{x} = z \\ \dot{t} = 1 \\ \dot{z} = 0 \end{cases} \qquad \begin{cases} x(r,0) = r \\ t(r,0) = 0 \\ z(r,0) = f(r). \end{cases} \tag{10.8}$$

We have

$$t(r,s) = s, \qquad z(r,s) = f(r) \tag{10.9}$$

so that

$$\begin{cases} \dot{x} = f(r) \\ x(r,0) = r \end{cases} \quad \Rightarrow \quad x(r,s) = r + f(r)s. \tag{10.10}$$

■

10.2 Shock Waves

Non-linear 1st order PDEs can exhibit a phenomenon called shock wave, which mathematically is a discontinuity in the solution, which propagates in time.

there are not so many to distill...

Lecture 33 (July 22)

Method of characteristics (continued)

Shock waves

Non-linear 1st order PDEs can exhibit a phenomenon called *shock wave*, which mathematically is a discontinuity in the solution, which propagates in time.

Let's look again at what happens to Burger's equation, with a Gaussian initial condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = e^{-x^2} & x \in \mathbb{R}, \end{cases} \tag{1}$$

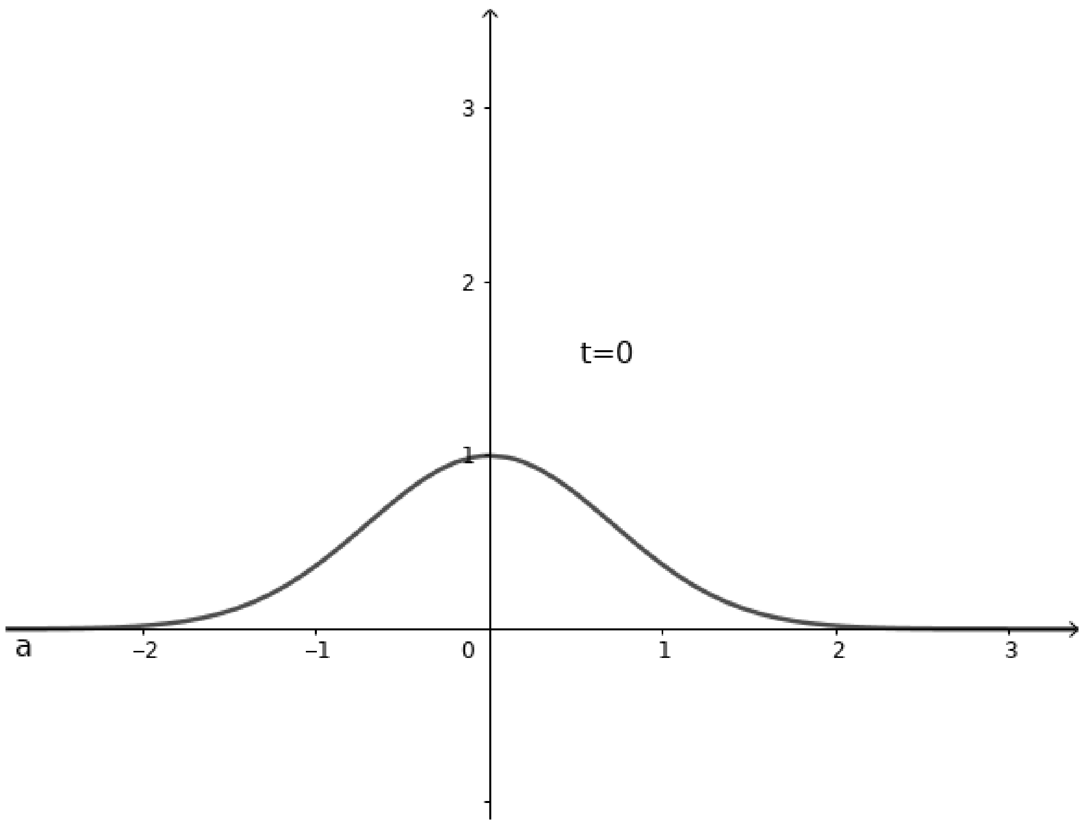
whose parametric surface is given by

$$\begin{cases} x(r,s) = r + e^{-r^2}s \\ t(r,s) = s \\ z(r,s) = e^{-r^2}. \end{cases} \tag{2}$$

Note that the parameter s is identified with time, so it has a physical interpretation; however, we cannot explicitly invert (at least not in terms of elementary functions). This is not a huge problem, since we can just graph $u(x,t)$ for each fixed t as the parametric curve

$$\{(r + e^{-r^2}t, e^{-r^2}) \mid r \in \mathbb{R}\}. \tag{3}$$

The real problem is that this curve cannot always be interpreted as the graph of a function! Look at the animation below: after a certain time $t_* \approx 1.2$ the function becomes multi-valued.



We say that at time t_* the solution breaks, or that a *shock* develops.

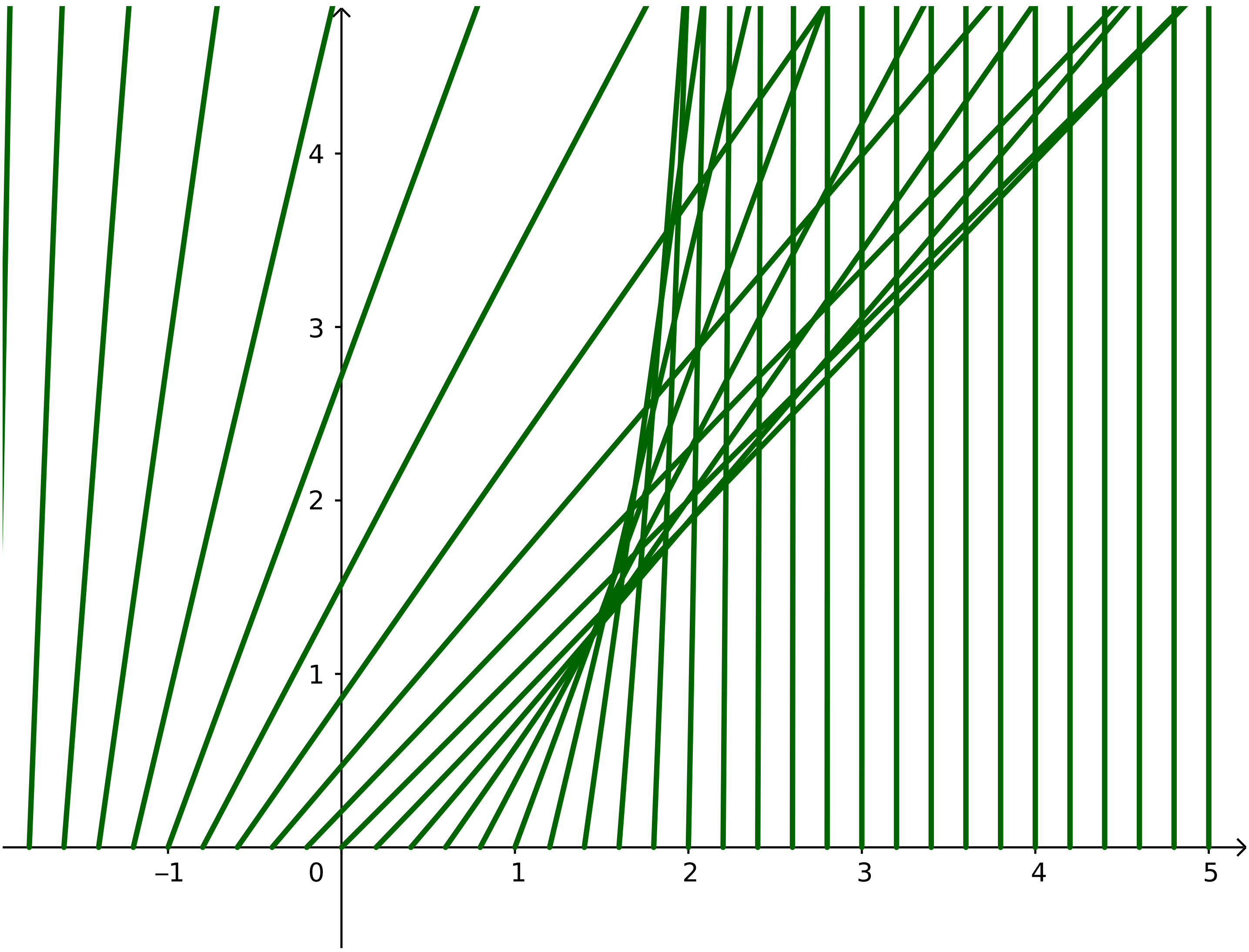
What causes the shock?

The shock is caused by the fact that the propagation speed is proportional to u: the tip of the Gaussian function in the initial data ($x = 0$) propagates faster than the points at $x > 0$, and eventually catches up.

Mathematically, what happens is that *the (projected) characteristics intersect*. This means that, if we look at the characteristic curves projected in the xt–plane

$$\{(r + e^{-r^2}t, t) \mid t \geq 0\}. \tag{4}$$

where r labels the curve, they are going to intersect (see image below).



Recall that the initial data (defined on the initial curve $t = 0$) propagates along the characteristics. Then we can see that at some point in time two different initial data points are propagated to the same point, and the solution breaks.

Identifying time and position of the initial shock

Figuring out what happens to the solution after the shock develops is not easy (we may talk a bit about it as extra material in the next lectures). For now, let's focus in finding the exact time t_* and position x_* at which the shock develops.

Let's consider the general case of Burger's equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 & x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) = f(r) & x \in \mathbb{R}, \end{cases} \tag{5}$$

and suppose that the characteristics intersect. Recall that

$$\begin{cases} x(r,t) = r + f(r)t \\ z(r,t) = f(r), \end{cases} \tag{6}$$

where we identified s with t.

Since the solution we get using the method of characteristics is smooth, when $u(x,t)$ (with t fixed) is multi–valued it must have vertical tangent at some points; the time t_* is the first t for which a vertical tangent appears.

To find out when the slope $\frac{\partial z}{\partial x}$ becomes infinite (vertical tangent) we use the following trick:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{\partial r}{\partial x} \tag{7}$$

$$1 = \frac{\partial x}{\partial x} = \frac{\partial r}{\partial x}\big(1 + f'(r)t\big) \tag{8}$$

so that

$$\frac{\partial z}{\partial x} = \frac{f'(r)}{1 + f'(r)t}. \tag{9}$$

Assuming that $f'(r)$ is well–defined for all r, the only way the slope can become infinite is if

$$t = -\frac{1}{f'(r)}. \tag{10}$$

Then t_* is the smallest positive value of $-\frac{1}{f'(r)}$, that is

$$t_* = \min\left\{-\frac{1}{f'(r)}\,\middle|\, r \in I_+\right\}, \quad I_+ = \{r \mid f'(r) < 0\}. \tag{11}$$

If r_* is the value of r at which

$$t_* = -\frac{1}{f'(r_*)} \tag{12}$$

the position at which the shock occurs is

$$x_* = r_* + f(r_*)\, t_*. \tag{13}$$

In the specific example of $f(r) = e^{-r^2}$, we have

$$f'(r) = -2re^{-r^2}, \qquad -\frac{1}{f'(r)} = \frac{e^{r^2}}{2r}, \qquad I_+ = \{r \mid r > 0\}. \tag{14}$$

To find the minimum of $g(r) = \frac{e^{r^2}}{2r}$ we look at the critical points. We have

$$g'(r) = \frac{(2r^2 - 1)\, e^{r^2}}{2r^2} \tag{15}$$

so that the only critical point with $r > 0$ is

$$r_* = \frac{1}{\sqrt{2}}, \tag{16}$$

which is a minimum since

$$g''(r_*) = \frac{(2r_*^4 - r_*^2 + 1)\, e^{r_*^2}}{r_*^3} = 2\sqrt{2e} > 0. \tag{17}$$

Therefore

$$t_* = -\frac{1}{f'(r_*)} = \frac{e^{1/2}}{2/\sqrt{2}} = \sqrt{\frac{e}{2}} \approx 1.17 \tag{18}$$

and

$$x_* = r_* + e^{-r_*^2}\, t_* = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\sqrt{\frac{e}{2}} = \sqrt{2}. \tag{19}$$

11. Assignments & Other

11.1 Physical Interpretation of PDEs

■ **Example 11.1** Consider IBVP

$$\begin{cases} \frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0 \\ u(0,t) = t \\ \frac{\partial u}{\partial x}(L,t) = 0 \\ u(x,0) = 0 \end{cases}$$

- $u(0,t)$ means that the temperature at the left side is proportional to time.
- $\frac{\partial u}{\partial x}(L,t) = 0$ means there is no flux at the right side. Insulated.
- $u(x,0) = 0$ means that the temp is zero everywhere at time $t = 0$.

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 5$$

The source term tells us that the heat energy is introduced at a constant rate into the rod by some heat source.

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = u^2$$

- u^2 should be a source term since it cannot come from the flux term of the conservation law $\left(\frac{\partial \phi}{\partial t}\right)$ since it has no derivatives w.r.t x.
- The source term tells us that the heat energy is introduced at a rate that is proportional to the temp squared.

■

11.2 Resonances

Physically, what causes resonance is that we move the membrane up and down at a freq close to one of the natural freqs of the membrane, creating a positive feedback effect.

11.3 general solution to BVP

The functions $M_k(x)$ form a complete orthogonal set (Fourier series) so any reasonable condition $u(x, 0) = f(x)$ can be satisfied by the solution from part (c).