Contents

♠ | Pre-mid

Review the proofs for fourier relations...

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2.1 Sampling Theorem

Ω is the angular velocity.

A function $f(t)$ is said to be bandlimited, or Ω-bandlimited, if there exists an $\Omega > 0$ such that

$$F(\omega) = 0 \qquad \text{for } |\omega| > \Omega$$

Nyquist freq: $\nu = \dfrac{\Omega}{2\pi}$

The Whittaker-Shannon Sampling Theorem : $f(t)$ Ω-bandlimited. Then $f = \mathcal{F}^{-1}F$ is completely determined at any $t \in \mathbb{R}$ by its values at $t_k = \frac{k\pi}{\Omega}$, $k = 0, \pm 1, \pm 2, \ldots$, as follows

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\Omega}\right) \frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi} = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\Omega}\right) \operatorname{sinc}\left(\frac{\Omega t}{\pi} - k\right)$$

The sampling freq is twice the bandwidth freq Ω

effects of undersampling

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one consequence of uncertainty principle: A function and its Fourier transform cannot both have finite support.

windowed FT, local freq of a signal

3.1 Wavelets and multiresolution analysis

3.1.1 Introduction

Detail or wavelet function

$$\psi(t) = \begin{cases} 1 & 0 \leq t < 1/2 \\ -1 & 1/2 \leq t < 1 \end{cases}$$

Let us introduce the following space of functions:

$$V_0 = \{f \in L^2(\mathbb{R}) : \; f(t) \text{ is constant over the interval } [k, k+1), \quad \forall k \in Z\}$$

and $\phi_{0k}(t) = I_{[k,k+1)}(t)$ spans V_0.

$$\phi_{0k}(t) = \phi(t-k)$$

We notice that the ϕ_{1k} are translated copies of ϕ_{10}. But what is more important is that they also are **dilated and translated** copies of the scaling function $\phi(t)$:

$$\phi_{1k} = \sqrt{2}\phi(2t-k)$$

they span V_1.

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4.1 Intro cont'd

Go backwards:

$$V_{-1} = \text{... constant over } [2k, 2k+1) = \text{span}\left\{\frac{1}{\sqrt{2}}\phi\left(\frac{t}{2} - k\right), \forall k \in \mathbb{Z}\right\} \cap L^2(\mathbb{R})$$

In general, for $J \in \mathbb{Z}$.

$$V_J = \{\text{...over } \left[\frac{k}{2^J}, \frac{k+1}{2^J}\right), \forall k \in \mathbb{Z}\}$$

and it is spanned by $2^{J/2}\phi(2^J t - k)$.

4.1.1 Nesting relation

$$\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots$$

4.1.2 Nesting relation $V_0 \subset V_1$

$$\phi(t) = \phi(2t) + \phi(2t-1), \qquad t \in [0, 1]$$

More mathematical:

$$\phi(t) \sum_{k \in \mathbb{Z}} h_k \phi_{1k}(t) = \sum_{k \in \mathbb{Z}} h_k \sqrt{2}\phi(2t - k)$$

latter one is called multiresolution analysis or refinement equation that is satisfied by a scaling function. It is sometimes simply called scaling equation. The nonzero coeff h_k are known as scaling coeff. For Haar system:

$$h_k = \begin{cases} \frac{1}{\sqrt{2}} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

4.2 wavalets

Using $V_0 \subset V_1$, we define $W_0 = V_0^\perp$.

So $V_1 = V_0 \oplus W_0$.

The function $\phi(t)$ is known at the **mother wavelet** of the Haar wavelet system, or simply the **Haar mother wavelet**.

4.2.1 Summary of major recent results

1. multiresolution analysis:
$$\phi(t) = \sum_{k\in\mathbb{Z}} h_k \sqrt{2}\phi(2t - k)$$
non-zero coeffs characterize a particular multiresolution analysis.
2. For each $j \in \mathbb{Z}$, the infinite set of functions
$$\phi_{jk}(t) = 2^{j/2}\phi(2^j t - k), \qquad k \in \mathbb{Z}$$
orthonormal basis for V_j.
3. nested
4. $V_1 = V_0 \oplus W_0$
5. **Haar wavelet function** $\psi(t)$ is a function in V_1 that is orthogonal to $\phi(t)$
$$\psi(t) = \phi(2t) - \phi(2t - 1)$$
More generally,
$$\psi(t) = \sum_{k\in\mathbb{Z}} g_k \sqrt{2}\phi(2t - k)$$
6. The integer translates of $\psi(t)$ form an orthonormal basis of W_0.
A consequence $V_1 = V_0 \oplus W_0$, we need two sets of functions: ϕ_{0k} and ψ_{0k}:
$$u(t) = \sum_{k\in\mathbb{Z}} a_k \phi_{0k}(t) + \sum_{k\in\mathbb{Z}} b_k \psi_{0k}(t)$$

4.2.2 Attention! Important summary

The function $f_1 \in V_1$ which is the best approximation of f in the space V_1 will admit the following expansion

$$f_1(t) = \sum_{k\in\mathbb{Z}} a_k \phi_{0k}(t) + \sum_{k\in\mathbb{Z}} b_k \psi_{0k}(t)$$

where

$$a_k = \langle f_1, \phi_{0k} \rangle, \quad b_k = \langle f_1, \psi_{0k} \rangle, \qquad k \in \mathbb{Z}$$

Also $f_1 \in V_1$:

$$f_1(t) = \sum_{k\in\mathbb{Z}} c_k \phi_{1k}(t)$$

where

$$c_k = \langle f_1, \phi_{1k} \rangle$$

4.2.3 Higher-order nestings $V_j \subset V_{j+1}$

$$V_{j+1} = V_j \oplus W_j$$

Then

$$\lim_{J\to\infty} V_J = L^2(\mathbb{R})$$

This means that

$$L^2(\mathbb{R}) = V_0 \oplus \left[\bigoplus_{j=0}^{\infty} W_j\right]$$

If we decompose V_0 more, then

$$L^2(\mathbb{R}) = \bigoplus_{j=0}^{\infty} W_j$$

Consequently, any function $f \in L^2(\mathbb{R})$ admits a unique expansion of the form

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} b_{jk} \psi_{jk}(t)$$

4.2.4 Connection to the (continuous) wavelet transform introduced earlier

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4.2.5 Special case: Haar wavelet expansions of functions on a finite interval

$$f(t) = f_0(t) + \sum_{j=1}^{\infty} w_j(t)$$

where $f_0(t) = a_{00}\phi_{00}(t)$. Detail functions, $w_j \in W_j$ are defined:

$$w_j(t) = \sum_{k=0}^{2^j-1} b_{jk}\psi_{jk}(t), \qquad j = 0, 1, 2, \dots$$

In **practical** situations, we deal with finite-dimensional.

$$f_j(t) = f_0(t) + \sum_{i=0}^{j-1} w_i(t) = a_{00}\phi_{00}(t) + \sum_{i=0}^{j-1}\sum_{k=0}^{2^i-1} b_{ik}\psi_{ik}(t)$$

Summary [1] a_{00} represents constant approx. $f_0 \in V_0$ to f. Using a_{00} and b_{00} produces f_1. Then adding b_{10} and b_{11} produces $f_2 \in V_2$.

4.2.6 Analysis and synthesis alg for wavelet expansions

Analysis/decomposition alg

$$f_1(t) = \sum_{k\in\mathbb{Z}} a_{1k}\sqrt{2}\phi(2t-k)$$

[1] last paragraph of page 318

and

$$f_1(t) = \sum_{k\in\mathbb{Z}} a_{0k}\phi(t-k) + \sum_{k\in\mathbb{Z}} b_{0k}\psi(t-k)$$

Now introduce $A_{jk} = 2^{j/2}a_{jk}, \qquad B_{jk} = 2^{j/2}b_{jk}$

Then two eqs above become:

$$f_1(t) = \sum_{k\in\mathbb{Z}} A_{1k}\phi(2t-k)$$

and

$$f_1(t) = \sum_{k\in\mathbb{Z}} A_{0k}\phi(t-k) + \sum_{k\in\mathbb{Z}} A_{0k}\psi(t-k)$$

After some calculations, we get

$$A_{0k} = \frac{1}{2}[A_{1,2k} + A_{1,2k+1}]$$
$$B_{0k} = \frac{1}{2}[A_{1,2k} - A_{1,2k+1}]$$

Given coeff for f_1, we compute coeff of $V_0 \oplus W_0$ decomposition of V_1.

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5.1 Analysis and synthesis alg for wavelet expansions cont'd

5.1.1 General Resolutions

$V_j = V_{j-1} \oplus W_{j-1}$

$$A_{j-1,k} = \frac{1}{2}[A_{j,2k} + A_{j,2k+1}]$$
$$B_{j-1,k} = \frac{1}{2}[A_{j,2k} - A_{j,2k+1}]$$

Reconstruction/synthesis

Step j Take $a_{j-1,k}$ coeff, along with b_{jk}. compute a_{jk}, $0 \leq k \leq 2^j - 1$.

5.2 Multi-re analysis: A general treatment

The collection $\{V_j\}$ is called a multi-re analysis with scaling function ϕ if the following conditions hold:

1. nesting
2. density: $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$

   This essentially states, in proper set-theoretic language, that $\lim_{j \to \infty} V_j = L^2(\mathbb{R})$
3. separation: $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$

4. scaling: $f(x) \in V_j \iff f(2x) \in V_{j+1}$

$$f(x) \in V_j \iff f(2^{-j}x) \in V_0$$

Subspaces V_j satisfying 1-4 are known as approximation spaces

5. orthonormal basis

Theorem If the support of the scaling function $\phi(x)$ is finite, then only a finite number of the coeff h_k can be nonzero.

Proof Suppose $\phi(x) = 0$ outside the interval $[-a, a]$, where $a > 0$ is finite. Also let $k_1 < k_2 < \ldots$ be an infinite seq of ints for which $h_{k_i} \neq 0$. Now suppose that $\phi(p) \neq 0$ for some $p \in [-a, a]$. Then from the scaling eqs,

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \sqrt{2}\phi(2x - k)$$

it follows that there will be nonzero contributions to RHS at the points $x_i \in \mathbb{R}$ defined by $2x_i - k_i = p$, $i = 1, 2, \ldots$, implying that the values $\phi(x_i)$ are nonzero. But a rearrangement yields $x_i = \frac{1}{2}(p + k_i)$, implying that $x_i \to \infty$ as $i \to \infty$. This contradicts the assumption that $\phi(x)$ is zero outside the interval $[-a, a]$. □

5.2.1 Wavelet spaces

$$W_0 = V_0^{\perp}$$

$$\psi(x) = \sum_{k\in\mathbb{Z}} g_k \phi_{1k} = \sum_{k\in\mathbb{Z}} g_k \sqrt{2}\phi(2x - k)$$

and

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \phi_{1k} = \sum_{k\in\mathbb{Z}} \sqrt{2}\phi(2x - k)$$

Inner product $= 0$

$$\langle \psi, \phi \rangle = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} g_k \bar{h}_l \langle \phi_{1k}, \phi_{1l} \rangle = \sum_{k\in\mathbb{Z}} g_k \bar{h}_k$$

It must $= 0$ by orthogonality.

cheap trick: Set $g_k = (-1)^k \bar{h}_{1-k}$

Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} g_k\bar{h}_k &= \ldots + g_{-1}\bar{h}_{-1} + g_0\bar{h}_0 + g_1\bar{h}_1 + g_2\bar{h}_2 + \ldots \\ &= \ldots - \bar{h}_2\bar{h}_{-1} + \bar{h}_1\bar{h}_0 - \bar{h}_0\bar{h}_1 + \bar{h}_2\bar{h}_{-1} + \ldots \\ &= 0\end{aligned}$$

Theorem For any $j \in \mathbb{Z}$, the set of functions $\{\psi_{jk} = 2^{j/2}\psi(2^j x - k)\}$ where

$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k \bar{h}_{1-k}\phi(2x-k)$$

forms an orthonormal basis of W_j.

♠ | Week 11

6.1 MRA: general (cont'd)

6.1.1 Wavelet spaces (cont'd)

From density property for MRA, we may write loosely: $\lim_{j\to\infty} V_j = L^2(\mathbb{R})$, so that the equation becomes

$$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus \dots$$

Then...

$$L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j$$

The **doubly indexed** set of functions $\{\psi_{jk} = 2^{j/2}\psi(2^j - k)\}, \quad j, k \in \mathbb{Z}$ forms an orthonormal basis in $L^2(\mathbb{R})$.

6.1.2 Synthesis and Analysis Alg for MRAs

Analysis

General decomposition:

$$V_j = V_{j-1} \oplus W_{j-1}$$

finer scale = coarser scale + detail.

In scaling function, replace x with $2^{j-1}x - l$, and $m = 2l + k$, we have

$$\phi(2^{j-1}x - l) = \sum_{m\in\mathbb{Z}} h_{m-2l}\sqrt{2}\phi(2^j x - m)$$

Analysis Express the coarser coeff $a_{j-1,k}, b_{j-1,k}$ in terms of finer coeff $a_{j,k}$.

Then calculate directly using inner product

$$a_{j-1,l} = \langle f, \phi_{j-1,l} \rangle$$

Synthesis express the finer coeff $a_{j,k}$ in terms of coarser coeff $a_{j-1,k}, b_{j-1,k}$.

6.1.3 Wavelets with compact support

very important!

Two ways:

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_k \phi_{1k}, \quad h_k = \langle \phi, \phi_{1k} \rangle$$

and

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_k \sqrt{2} \phi(2x - k)$$

Aforementioned theorem: finite support of ϕ implies finite number of nonzero coeff h_k.

some conditions that must be satisfied by the scaling coeff h_k for $\phi(x)$ to have compact support.

1. Finite energy (squared L^2 norm)

$$\langle \phi, \phi \rangle = \sum_k |h_k|^2 = 1$$

2. Finite L^1 norm.

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_k \sqrt{2} \phi(2x - k) \text{ [1]}$$

Since in $L^2[a,b]$, then $L^1[a,b]$

$$\left| \int_{\mathbb{R}} \phi(x) \, dx \right| \leq \int_{\mathbb{R}} |\phi(x)| dx < \infty$$

implying that the integral on the left exists. Now integrate both sides of scaling equation:

[1] An important eq which I will use later

$$\int_{\mathbb{R}} \phi(x)\ dx = \sum_k h_k \sqrt{2} \int_{\mathbb{R}} \phi(2x-k)\ dx$$

For each k, we have this by let $s = 2x - k$,

$$\int_{\mathbb{R}} \phi(2x-k)\ dx = \frac{1}{2} \int_{\mathbb{R}} \phi(s)\ ds$$

Then sub into prev eq,

$$\int_{\mathbb{R}} \phi(x) dx = \sum_k h_k \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \phi(x) dx$$

Since non-zero integral, then we have

$$\sum_k h_k = \sqrt{2}$$

3. Generalized orthogonality

$$\langle \phi(x), \phi(x-p) \rangle = 2 \sum_k \sum_l h_k\ h_l\ \ \langle \phi(2x-k), \phi(2x-2p-l) \rangle = \delta_{0p}$$

Setting $k = 2p+l$, we make the inner product inside to be $\frac{1}{2}$ since $\langle \phi(2x-k), \phi(2x-2p-l) \rangle = \frac{1}{2}\delta_{k,2p+l}$.
So we have final result:

$$\sum_k h_k\ h_{k-2p} = \delta_{0p}$$

Important sequence the length of the seq of nonzero h_k must be even.

4. $\sum$ even-indexed $=$ $\sum$ odd-indexed

$$\sum_k h_{2k} = \sum_k h_{2k+1} = \frac{1}{\sqrt{2}}$$

Proof Define

$$K_0 = \sum_k h_{2k}, \qquad K_1 = \sum_k h_{2k+1}$$

use orthogonality:

$$\sum_k h_k h_{k+2n} = \delta_{0n}$$

Sum both sides over n:

$$\sum_n \sum_k h_k h_{k+2n} = \sum_n \delta_{0n} = 1$$

Then split the sum:

$$\begin{aligned}
\sum_n \sum_k h_k h_{k+2n} &= \sum_n \left[\sum_k h_{2k} h_{2k+2n} + \sum_k h_{2k+1} h_{2k+1+2n} \right] \\
&= \sum_k \left[\sum_n h_{2k+2n} \right] h_{2k} + \sum_k \left[\sum_n h_{2k+2n+1} \right] h_{2k+1} \\
&= \sum_k \left[\sum_n h_{2(k+n)} \right] h_{2k} + \sum_k \left[\sum_n h_{2(k+n)+1} \right] h_{2k+1} \\
&= K_0 \sum_k h_{2k} + K_1 \sum_k h_{2k+1} \\
&= K_0^2 + K_1^2 \\
&= 1
\end{aligned}$$

and $K_0 + K_1 = \sqrt{2}$, then $K_0 = K_1 = \frac{1}{\sqrt{2}}$

6.1.4 Relating the support of $\phi(x)$ to nonzero h_k coeff

Theorem If $\phi(x)$ finite support on $[N_1, N_2]$, both $\mathbb{Z}$, then $h_k = 0$ for both $k > N_2$ and $k < N_1$.

In this case, h_k are said to have compact support in $[N_1, N_2]$.

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? Is proof required?

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Proof the support of the function $\phi(2x - k)$ must lie inside the interval determined by the ineq:

$$N_1 \leq 2x - k \leq N_2 \implies \frac{1}{2}(N_1 + k) \leq x \leq \frac{1}{2}(N_2 + k)$$

Also, interval must lie inside $[N_1, N_2]$:

$$[\frac{1}{2}(N_1 + k), \frac{1}{2}(N_2 + k)] \subseteq [N_1, N_2] \implies N_1 \leq k \leq N_2$$

Theorem If $\phi(x)$ finite $[N_1, N_2]$, then $\psi(x)$ compact on $[\frac{1}{2}(N_1 - N_2 + 1), \frac{1}{2}(N_2 - N_1 + 1)]$.

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? Is proof required?

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♠ | Week 12

7.1 MRA: general treatment (cont'd)

7.1.1 $\psi(x)$ its support, and vanishing moments

$$\begin{aligned}\int_{\mathbb{R}} \psi(x)dx &= \sum_k g_k\sqrt{2}\int_{\mathbb{R}} \phi(2x-k)dx \\ &= \sum_k (-1)^k h_{1-k}\sqrt{2}\int_{\mathbb{R}} \phi(2x-k)dx\end{aligned}$$

for any $k \in \mathbb{Z}$:

$$\begin{aligned}\int_{\mathbb{R}} \phi(2x-k)dx &= \frac{1}{2}\int_{\mathbb{R}} \phi(x)dx \quad (s = 2x-k, ds = 2dx, \text{ etc. }) \\ &= \frac{1}{2}M\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}} \psi(x)dx &= \frac{1}{\sqrt{2}}M\sum_k (-1)^k h_{1-k} \\ &= -\frac{1}{\sqrt{2}}M\sum_l (-1)^l h_l \quad (l = 1-k \Rightarrow k = 1-l, \text{ etc. }) \\ &= -\frac{1}{\sqrt{2}}M\left[\sum_k h_{2k} - \sum_k h_{2k+1}\right] \\ &= 0\end{aligned}$$

Let us now assume that $f(x)$ is a polynomial:

$$f(x) = \sum_{k=0}^{n} c_k x^k$$

over interval I which contains the domain of support of D of a wavelet ψ.

$$\begin{aligned} b_{00} &= \int_I f(x)\psi(x)dx \\ &= \int_D \sum_{k=0}^{n} c_k x^k \psi(x)dx \\ &= \sum_{k=0}^{n} c_k \int_D x^k \psi(x)dx \\ &= \sum_{k=0}^{n} c_k m_k \end{aligned}$$

where

$$m_k = \int_{\mathbb{R}} x^k \psi(x)\ dx \qquad k \geq 0$$

is know as kth moment of the wavelet function. If $m_k = 0$ for $0 \leq k \leq n$ then $b_{00} = 0$.

Defn ψ has M vanishing moments:

$$m_k = \int_{\mathbb{R}} x^k \psi(x)dx = 0, \quad k = 0, 1, 2, \cdots, M-1$$

Implications: $M - 1$ vanishing moment, we have an upper bound to b_{jk}

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? How much should we know about this?

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7.1.2 Vanishing moments and the approximation of functions

Theorem V_j with scaling function ϕ. ψ has M vanishing moments. Best approx in V_j in L^2 sense:

$$f_j = \sum_{k\in\mathbb{Z}} \langle f, \phi_{jk} \rangle \, \phi_{jk}$$

Then L^2 error has an bound

$$\|f - f_j\|_2 \leq C2^{-jM}$$

7.2 MRA and Fourier transform

Two theorems here (not responsible)

7.2.1 Fourier transforms and vanishing moments of wavelets

$$\begin{aligned} F^{(n)}(\omega) &= \frac{d^n}{d\omega^n}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\omega x}dx\right] \\ &= (-i)^n \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^n f(x)e^{-i\omega x}dx \end{aligned}$$

From this result,

$$\Psi^{(k)}(0) = 0 \quad \text{implies that } m_k = 0$$

Therefore, if

$$\Psi^{(k)} = 0, \quad 0 \leq k \leq M-1$$

then $\psi(x)$ has M vanishing moments.