Introduction to Theoretical Mechanics

AMATH 271

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LaTeXed by Sibelius Peng

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Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of AMATH 271 during Fall 2019 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

These notes are my learning notes from the textbook: *Classical Mechanics*, by J. R. Taylor, University Science Books, 2005, along with the course outline of AMATH 271.

For any questions, send me an email via https://notes.sibeliusp.com/contact/.

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1

Newton's Laws of Motion

1.1 Classical Mechanics

Mechanics is the study of how things move: how a skier moves down the slope. or how an electron moves around the nucleus of an atom.

Relativistic mechanics to describe very high speed motions and quantum mechanics to describe the motion of microscopic particles.

1.2 Space and Time

space, time, mass and force.

Space

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \tag{1.1}$$

It is sometimes convenient to be able to abbr (1.1) by writing simply

$$\mathbf{r} = (x, y, z) \tag{1.2}$$

Vector Operations

Differentiation of Vectors

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \tag{1.3}$$

where

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \tag{1.4}$$

If $f(t)$ is a scalar:

$$\frac{d}{dt}(f\mathbf{r}) = f\frac{d\mathbf{r}}{dt} + \frac{df}{dt}\mathbf{r} \tag{1.5}$$

Time

Reference Frames

Reference Frame

Almost every problem in classical mechanics involves a choice (explicit or implicit) of a reference frame, that is, a choice of spatial origin and axes to label positions and a choice of temporal origin to measure times.

In certain special frames, called inertial frames, the basic laws hold true in their standard, simple form. If a second frame is accelerating or rotating relative to an inertial frame, then this second frame is noninertial, and the basic laws in particular, Newton's laws - do not hold in their standard form in this second frame.

Inertial Frames

An inertial frame is any reference frame in which Newton's first law holds, that is, a nonaccelerating, nonrotating frame.

1.3 Mass and Force

The mass of an object characterizes the object's inertia - its resistance to being accelerated.

1.4 Newton's First and Second Laws; Inertial Frames

In this chapter, he is going to discuss Newton's laws as they apply to a **point mass**.

Newton's First Law (the Law of Inertia)

In the absence of forces, a particle moves with constant velocity $\mathbf{v}$.

and

Newton's Second Law

For any particle of mass m, the net force $\mathbf{F}$ on the particle is always equal to the mass m times the particle's acceleration:

$$\mathbf{F} = m\mathbf{a}$$

$$\begin{aligned} \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \equiv \dot{\mathbf{v}} \\ &= \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \equiv \ddot{\mathbf{r}} \end{aligned}$$

Momentum is defined as

$$\mathbf{p} = m\mathbf{v}$$

In classical mechanics, we take for granted that the mass m of a particle never changes, so that

$$\dot{\mathbf{p}} = m\dot{\mathbf{v}} = m\mathbf{a}$$

Thus the second law can be rephrased to say that

$$\mathbf{F} = \dot{\mathbf{p}}$$

Differential Equations

Consider Newton's second law for a particle confined to move along the x axis and subject to a constant force F_o,

$$\ddot{x}(t) = \frac{F_o}{m}$$

Inertial Frames

$\mathcal{S}$ denotes a reference frame.

Validity of the First Two Laws

1.5 The Third Law and Conservation of Momentum

Newton's Third Law

If object 1 exerts a force $\mathbf{F}_{21}$ on object 2, then object 2 always exerts a reaction force $\mathbf{F}_{12}$ on object 1 given by

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

In absence of external forces, the total momentum of our two-particle system is constant - a result called the principle of conservation of momentum.

Multiparticle Systems

$$(\text{net force on particle } \alpha) = \mathbf{F}_\alpha = \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}}$$

Principle of Conservation of Momentum

If the net external force $\mathbf{F}^{ext}$ on an N-particle system is zero, the system's total momentum $\mathbf{P}$ is constant.

Validity of Newton's Third Law

Therefore, the third law cannot be valid once relativity becomes important.

1.6 Newton's Second Law in Cartesian Coordinates

Of Newton's three laws, the one we actually use the most is the second, which is often described as the equation of motion.

$$\mathbf{F} = m\ddot{\mathbf{r}} \iff \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases}$$

1.7 Two-Dimensional Polar Coordinates

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$$

Since the two unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are perpendicular in our 2d space, any vector can be expanded in terms of them.

$$\mathbf{F} = F_r \hat{\boldsymbol{r}} + F_\phi \hat{\boldsymbol{\phi}}$$

Tedious calculations... (See page 29 of the textbook for details)

Having calculated the acceleration, we can finally write down Newton's second law in terms of polar coordinates:

$$\mathbf{F} = m\mathbf{a} \iff \begin{cases} F_r = m(\ddot{r} - r\ddot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases}$$

... and Newton's Second Law in Cylindrical Polar:

$$\begin{cases} F_r = m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\ F_z = m\ddot{z} \end{cases}$$

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2

Projectiles and Charged Particles

2.1 Air Resistance

Let us begin by survey some of basic properties of the resistance force, or **drag**, $\mathbf{f}$ of the air, or other medium, through which an object is moving.

Yous should, however, be aware that there are situations where it[1] is not certainly true: The force of the air on an airplane wing has a large sideways component, called the **lift**, without which no airplanes could fly.

$$\mathbf{f} = -f(v)\hat{\boldsymbol{v}}$$

At lower speeds it is often a good approximation to write

$$f(v) = bv + cv^2 = f_{lin} + f_{quad}$$

where they stand for the linear and quadratic terms respectively,

$$f_{lin} = bv \qquad \text{and} \qquad f_{quad} = cv^2$$

- The linear term, arises from the viscous drag of the medium and is generally proportional to the viscosity of the medium and the linear size of the projectile.
- The quadratic term, arises from the projectile's having to accelerate the mass of air with which it is continually colliding; it is proportional to the density of the medium and the cross-sectional area of the projectile.

Reynolds number

is an important parameter that features prominently in more advanced treatments of motion in fluids. Thus one can relate the ratio f_{quad}/f_{lin} to the fundamental

[1] The direction of the force due to motion through the air is opposite to the velocity $\mathbf{v}$

parameters η, the viscosity, and ϱ, the density of the fluid. The result is that the ratio is roughly the same order of magnitude as the dimensionless number $R = Dv\varrho/\eta$, called the **Reynolds number**. Thus a compact and general way to summarize the foregoing discussion is to say that the quadratic drag is dominant when R is large, whereas the linear drag dominates when R is small.

2.2 Linear Air Resistance

$$m\ddot{\mathbf{r}} = m\mathbf{g} - \mathbf{v}$$

Since neither of the forces depends on $\boldsymbol{r}$, then

$$m\dot{\mathbf{v}} = m\mathbf{g} - b\mathbf{v}$$

Two separate equations on x and y direction respectively (one for v_x one for v_y).

But if the drag force is quadratic, we then have two coupled DEs are much harder to solve than the uncoupled equations of the linear case since each equation involves both of the variables v_x and v_y.

Horizontal Motion with Linear Drag

$$\dot{v_x} = -kv_x$$

Then

$$v_x(t) = Ae^{-kt} = v_{xo}e^{-kt} = e^{-t/\tau}$$

where

$$\tau = 1/k = m/b$$

So

$$x(t) = x_\infty(1 - e^{-t/\tau})$$

where

$$x_\infty = v_{xo}\tau$$

Vertical Motion with Linear Drag

$$m\dot{v}_y = mg - bv_y$$

Terminal speed:

$$v_{ter} = \frac{mg}{b}$$

Then solve the DE above, we get

$$v_y(t) = v_{yo}e^{-t/\tau} + v_{ter}(1 - e^{-t/\tau})$$

Then integrate it over t we get $y(t)$

$$y(t) = v_{ter}t + (v_{yo} - v_{ter})\tau(1 - e^{-t/\tau})$$

2.3 Trajectory and Range in a Linear Medium

Combine the results, then we get

$$x(t) = \dots, \quad y(t) = \dots$$

You can eliminate t, then we get

$$y = \frac{v_{yo} + v_{ter}}{v_{xo}}x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right)$$

2.4 Quadratic Air Resistance

$$\boldsymbol{f} = -cv^2\hat{\boldsymbol{v}}$$

Horizontal

$$m\frac{dv}{dt} = -cv^2$$

use separation of variables

Vertical

$$m\dot{v} = mg - cv^2$$

then

$$v_{ter} = \sqrt{\frac{mg}{c}}$$

then

$$v = v_{ter}\tanh\left(\frac{gt}{v_{ter}}\right)$$

and

$$y = \frac{v_{ter}^2}{g}\ln\left(\cosh\left(\frac{gt}{v_{ter}}\right)\right)$$

Both

$$m\ddot{\mathbf{r}} = m\mathbf{g} - cv\mathbf{v}$$

resolves into horizontal and vertical components: ...

which can't be solved analytically. The only way is numerically.

2.5 Motion of a Charge in a Uniform Magnetic Field

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

then

$$\begin{cases} m\dot{v}_x = qBv_y \\ m\dot{v}_y = -qBv_x \\ m\dot{v}_z = 0 \end{cases}$$

Cyclotron frequency:

$$\omega = \frac{qB}{m}$$

Then

$$\dot{v}_x = \omega v_y$$
$$\dot{v}_y = -\omega v_x$$

Let's define a complex number

$$\eta = v_x + iv_y$$

Then

$$\dot{\eta} = -i\omega\eta$$

then

$$\eta = Ae^{-i\omega t}$$

2.6 Complex Exponentials

Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

2.7 Solution for the Charge in a B Field

$$z(t) = z_o + v_{zo}t$$

and the motion of x and y is most easily found by introducing another complex number

$$\xi = x + iy$$

In the cyclotron, a device for accelerating charged particles to high energies, the particles are trapped in circular orbits in this way. They are slowly accelerated by the judiciously timed application of an electric field. The angular frequency of the orbit is, of course, $\omega = qB/m$. The radius of the orbit is

$$r = \frac{v}{\omega} = \frac{mv}{qB} = \frac{p}{qB}$$

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3

Momentum and Angular Momentum

3.1 Conservation of Momentum

Principle of Conservation of Momentum

If the net external force $\mathbf{F}^{ext}$ on an N-particle system is zero, the system's total mechanical momentum $\mathbf{P} = \sum m_\alpha \mathbf{v}_\alpha$ is constant.

3.2 Rockets

A beautiful example of the use of momentum conservation is the analysis of rocket propulsion.

Equation of Motion for a Rocket

$$m\dot{v} = -\dot{m}v_{ex} + F^{ext}$$

where $-\dot{m}$ is the rate at which the rocket's engine is ejecting mass.

3.3 The Center of Mass

center of mass

Let us consider a group of N particles, $\alpha = 1, \ldots, N$ with masses m_α and positions $\mathbf{r}_\alpha$ measured relative to an origin O. The center of mass (or CM) of this system is defined to be the position (relative to the same origin O)

$$\mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha$$

We can now write the total momentum $\mathbf{P}$ of any N-particle system in terms of the system's CM as follows:

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{R}}$$

One important point to bear in mind is that when the mass in a body is distributed continuously, the sum goes over to an integral:

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\, dm = \frac{1}{M}\int \varrho\mathbf{r}\, dV$$

3.4 Angular Momentum for a Single Particle

angular momentum

The angular momentum $\boldsymbol{\ell}$ of a single particle is defined as the vector

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$$

Remark:

Strictly speaking, refer $\boldsymbol{\ell}$ as the angular momentum relative to O.

The time rate of change of $\boldsymbol{\ell}$ is easily found:

$$\dot{\boldsymbol{\ell}} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = (\dot{\mathbf{r}} \times \mathbf{p}) + (\mathbf{r} \times \dot{\mathbf{p}})$$

Remark:

The first term on the right, $\mathbf{p}$ can be replaced by $m\dot{\mathbf{r}}$, and it's zero since two vectors are parallel, In the second term, replace $\dot{\mathbf{p}}$ by the net force $\mathbf{F}$ on the particle, then we get:

$$\dot{\boldsymbol{\ell}} = \mathbf{r} \times \mathbf{F} \equiv \boldsymbol{\Gamma}$$

Here $\boldsymbol{\Gamma}$ denotes the net torque about O on the particle.

Kepler's Second Law

Kepler's Second Law

As each planet moves around the sun, a line drawn from the planet to the sun sweeps out equal areas in equal times.

Proof:

The area of the triangle is

$$dA = \frac{1}{2}|\mathbf{r} \times \mathbf{v}\ dt|$$

Replacing $\mathbf{v}$ by $\mathbf{p}/m$ and dividing both sides by dt, we find that

$$\frac{dA}{dt} = \frac{1}{2m}|\mathbf{r} \times \mathbf{p}| = \frac{1}{2m}\ell$$

where ℓ denotes the magnitude of the angular momentum $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$. Thus dA/dt is constant. □

3.5 Angular Momentum for Several Particles

total angular momentum

$$\mathbf{L} = \sum_{\alpha=1}^{N} \boldsymbol{\ell}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha$$

By differentiating w.r.t t, we get

$$\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{ext}$$

In particular, if the net external torque is zero, we have the

Principle of Conservation of Angular Momentum

If the net external torque on an N-particle system is zero, the system's total angular momentum $\mathbf{L}$ is constant.

The Moment of Inertia

Specifically, if we take the axis of rotation to be the z axis, then L_z, the z component of angular momentum, is just $L_z = I\omega$, where I is the **moment of inertia** of the body for the given axis, and ω is the angular velocity of rotation.

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4

Energy

4.1 Kinetic Energy and Work

kinetic energy

or KE

$$T = \frac{1}{2}mv^2$$

$$\frac{dT}{dt} = \frac{1}{2}m\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}m(\dot{\mathbf{v}}\cdot\mathbf{v}+\mathbf{v}\cdot\dot{\mathbf{v}}) = m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}$$
$$\implies dT = \mathbf{F}\cdot d\mathbf{r}$$

Remark:

The expression on the right is defined to be the **work done by the force F** in the displacement $d\mathbf{r}$. Thus we have proved the **work-KE theorem**:

Theorem 4.1: Work-KE theorem

The change in the particle's kinetic energy between two neighbouring points on its path is equal to the work done by the net force as it moves between the two points.

4.2 Potential Energy and Conservative Forces

The gravitational force of the sun on a planet (position $\mathbf{r}$ relative to the sum):

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{r^2}\hat{\boldsymbol{r}}$$

Conditions for a Force to be Conservative

A force $\mathbf{F}$ acting on a particle is conservative if and only if it satisfies two conditions:

1. $\mathbf{F}$ depends only on the particle's position $\mathbf{r}$ (and not on the velocity $\mathbf{v}$, or the time t, or any other variable); that is $\mathbf{F} = \mathbf{F}(\mathbf{r})$.
2. For any two points 1 and 2, the work $W(1 \to 2)$ done by $\mathbf{F}$ is the same for all paths between 1 and 2.

Remark:

The reason for the name "conservative" and for the importance of the concept is this: If all forces on an object are conservative, we can define a quantity called the potential energy (or just PE), denoted $U(\mathbf{r})$, a function only of position, with the property that the total mechanical energy

$$E = \text{KE} + \text{PE} = T + U(\mathbf{r})$$

is constant; that is E is conserved.

potential energy

at an arbitrary point $\mathbf{r}$, to be

$$U(\mathbf{r}) = -W(\mathbf{r}_o \to \mathbf{r}) \equiv \int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

Several Forces

The change in the mass's kinetic energy is

$$\Delta T = W_{grav} + W_{spr} = -(\Delta U_{grav} + \Delta U_{spr})$$

Nonconservative Forces

$$\Delta T = W = W_{cons} + W_{nc}$$
$$\Delta E \equiv \Delta(T + U) = W_{nc}$$

4.3 Force as the Gradient of Potential Energy

We have seen the PE $U(\mathbf{r})$ corresponding to a force $\mathbf{F}$ can be expressed as an integral of $\mathbf{F}$. This suggests that we should be able to write $\mathbf{F}$ as some kind of derivative of

$U(\mathbf{r})$.

$$\mathbf{F} = -\boldsymbol{\nabla} U$$

Remark:

> The important relation gives us the force $\mathbf{F}$ in terms of derivatives of U. When a force $\mathbf{F}$ can be expressed in the form above, we say that $\mathbf{F}$ is **derivable from a potential energy**.

$$\boldsymbol{\nabla} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$$

$$df = \boldsymbol{\nabla} f \cdot d\mathbf{r}$$

4.4 The Second Condition that F be Conservative

There is a simple test, which can be quickly applied to any force that is given in analytic form. This test involves another of the basic concepts of vector calculus, this time the so-called *curl* of a vector.

It can be shown via Stokes's theorem that a force $\mathbf{F}$ has the desired property, that the work it does is independent of path, if and only if

$$\boldsymbol{\nabla} \times \mathbf{F} = 0$$

everywhere.

4.5 Time-Dependent Potential Energy

We sometimes have occasion to study a force $\mathbf{F}(\mathbf{r}, t)$ that satisfies the second condition to be conservative ($\boldsymbol{\nabla} \times \mathbf{F} = 0$), but, because it is time-dependent, does not satisfy the first condition.

4.6 Energy for Linear One-Dimensional Systems

The remarkable feature of one-dimensional systems is that the first condition already guarantees the second, so the latter is superfluous.

A second useful feature of one-dimensional systems is that with one independent variable (x) we can plot the potential energy $U(x)$, and, as we shall see, this makes it easy to visualize the behavior of the system.

$$U = \frac{1}{2}kx^2$$

for any spring obeying Hooke's Law.

A third remarkable feature of one-dimensional conservative systems is that we can – at least in principle – use the conservation of energy to obtain a complete solution of the motion, that is , to find the position x as a function of t.

We can solve for $T = \frac{1}{2}m\dot{x}^2 = E - U(x)$ and hence for the velocity $\dot{x}$ as a function of x:

$$\dot{x}(x) = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

4.7 Curvilinear One-Dimensional Systems

There are other systems that can equally be said to be one-dimensional, inasmuch as their position is specified by a single number.

4.8 Central Forces

If we take the force center to be the origin, a central force has the form

$$\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\hat{\boldsymbol{r}}$$

where $f(\mathbf{r})$ gives the magnitude of the force.

The Coulomb force has two additional properties not shared by all central forces:

1. conservative
2. spherically symmetric or rotationally invariant.

A compact way to express the second property is $f(\mathbf{r}) = f(r)$.

A remarkable feature of central forces is that the two properties just mentioned always go together. The most direct proofs involve the use of spherical polar coordinates.

4.9 Energy of Interaction of Two Particles

The force $\mathbf{F}_{12}$:

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Any isolated system must be translationally invariant.

4.10 The Energy of a Multiparticle System

kinetic energy T

$$T = T_1 + T_2 + T_3 + T_4$$

We can prove in general

$$-\boldsymbol{\nabla}_\alpha U = (\text{net force on particle } \alpha)$$

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5

Oscillations

5.1 Hooke's Law

$$F_x(x) = -kx$$

$$U(x) = \frac{1}{2}kx^2$$

5.2 Simple Harmonic Motion

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$$

where I have introduced the constant

$$\omega = \sqrt{\frac{k}{m}}$$

Exponential Solutions

$$x(t) = C_1 e^{i\omega t} + C - 2e^{-i\omega t}$$

superposition principle

Sine and Cosine Solutions

$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$$

where

$$B_1 = C_1 + C_2 \qquad \text{and} \qquad B_2 = i(C_1 - C_2)$$

The Phase-Shifted Cosine Slution

$$A = \sqrt{B_1^2 + B_2^2}$$

so

$$x(t) = A\cos(\omega t - \delta)$$

Solution as the Real Part of a Complex Exponential

$$x(t) = \mathrm{Re}\, Ce^{i\omega t} = \mathrm{Re}\, Ae^{i(\omega t - \delta)}$$

Energy Considerations

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t - \delta)$$

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t - \delta) = \frac{1}{2}kA^2\sin^2(\omega t - \delta)$$

Thus

$$E = T + U = \frac{1}{2}kA^2$$

5.3 Two-Dimensional Oscillators

In two or three dimensions, the possibilities for oscillations are considerably richer than in one dimension. The simplest possibility is the so-called **isotropic harmonic oscillator**, for which the restoring force is proportional to the displacement from equilibrium, with the same constant of proportionality in all directions:

$$\mathbf{F} = -k\mathbf{r}$$

In the anisotrophic oscillator, the components of the restoring force are proportional to the components of the displacement, but with different constants of proportionality.

5.4 Damped Oscillations

$$m\ddot{x} + b\dot{x} + kx = 0$$

ω_0: natural freq

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

For further discussions, please consult the book or AMATH 251 notes.

5.5 Driven Damped Oscillations

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

ω_0: natural freq

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

For further discussions, please consult the book or AMATH 251 notes.

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

5.6 Resonance

$$\begin{aligned}
\omega_0 &= \sqrt{k/m} = \text{ natural frequency of undamped oscillator} \\
\omega_1 &= \sqrt{\omega_o^2 - \beta^2} = \text{ frequency of damped oscillator} \\
\omega &= \text{ frequency of driving force} \\
\omega_2 &= \sqrt{\omega_0^2 - 2\beta^2} = \text{ value of } \omega \text{ at which response is maximum.}
\end{aligned}$$

In any case, the maximum amplitude of the driven oscillations is found by putting $\omega_0 \approx \omega$ to give

$$A_{max} \approx \frac{f_o}{2\beta\omega_o}$$

Width of The Resonance; the Q factor

We can define the width (or full width at half maximum or **FWHM**) as the interval between two points where A^2 is equal to half its maximum height.

$$FWHM \approx 2\beta$$

For many purposes, we want a very sharp resonances, so it is common practice to define a **quality factor** Q as the reciprocal of this ratio:

$$Q = \frac{\omega_0}{2\beta}$$

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6

Calculus of Variations

6.1 Two Examples

The calculus of variations involves finding the minimum or maximum of a quantity that is expressible as an integral. Two examples: The shortest path between two points and Fermat's Principle.

Since our concern is how infinitesimal variations of a path change an integral, the subject is called the calculus of variations. For the same reason, the methods we shall develop are called variational methods.

6.2 The Euler-Lagrange Equation

We have an integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), z]dx$$

where $y(x)$ is an as-yet unknown curve joining two points (x_1, y_1) and (x_2, y_2), which is

$$y(x_1) = y_1 \qquad \text{and} \qquad y(x_2) = y_2$$

Remark:

> To be definite, we shall suppose that we wish to find a minimum.
>
> Notice that the function f is a function of three variables $f = f(y, y', x)$, but because the integral follows the path $y = y(x)$ the integrand $f[y(x), y'(x), x]$ is actually a function of just the one variable x.

Let us denote the correct solution to our problem by $y = y(x)$. It is convenient to write the "wrong" curve $Y(x)$ as

$$Y(x) = y(x) + \eta(x)$$

where $\eta(x)$ is just the difference between the wrong $Y(x)$ and the right $y(x)$. Since $Y(x)$ must pass the endpoints 1 and 2, $\eta(x)$ must satisfy

$$\eta(x_1) = \eta(x_2) = 0$$

There are infinitely many choices for $\eta(x)$.

The integral S taken along the wrong curve $Y(x)$ must be larger than that along the right curve $y(x)$, no matter how close the former is close to the latter. To express this requirement, I shall introduce a parameter α and redefine $Y(x)$ to be

$$Y(x) = y(x) + \alpha\eta(x)$$

Denote the integral by $S(\alpha)$. Minimum is achieved at $\alpha = 0$. To ensure this, we must just check that the derivative is zero when $\alpha = 0$.

$$\begin{aligned} S(\alpha) &= \int_{x_1}^{x_2} f(Y, Y', x)dx \\ &= \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x)dx \end{aligned}$$

Now we differentiate it with respect to α. First, evaluate $\frac{\partial f}{\partial \alpha}$:

$$\frac{\partial f(y + \alpha\eta, y' + \alpha\eta', x)}{\partial \alpha} = \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}$$

and for $dS/d\alpha$ (which has to be zero)

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha}dx = \int_{x_1}^{x_2} \left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right) dx = 0$$

Use IBP, the second term:

$$\int_{x_1}^{x_2} \eta'(x)\frac{\partial f}{\partial y'}dx = \left[\eta(x)\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx = -\int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

Then

$$\int_{x_1}^{x_2} \eta(x)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx = 0$$

This condition must be satisfied by any choice of the function $\eta(x)$. Therefore, the factor in parentheses must be zero:

Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

for all x (in the relevant interval $[x_1, x_2]$).

6.3 Applications of E-L equation

Example: **Shortest Path between Two Points**

> ... Then $y' =$ constant. Thus $y(x) = mx+b$, and we have proved that the shortest path between two points is a straight line.

Example: **The Brachistochrone**

> A famous problem in the calculus of variations is this: Given two points 1 and 2, with 1 higher above the ground, in what shape should we build a frictionless roller coaster track so that a car released from point 1 will reach point 2 in the shortest possible time? This problem is called the brachistochrone problem, from the Greek words brachistos meaning “shortest” and chronos meaning “time.”

Maximum and Minimum vs. Stationary

The E-L equation guarantees only to give a path for which the original integral is stationary.

some discussion...

However, it should be clear that, in general, deciding what sort of stationary path the E-L equation has given us can be tricky.

Fortunately for us, these questions are irrelevant for our purposes. We shall find that for the applications in mechanics all that matters is that we have a path which makes a certain integral stationary. It simply doesn't matter whether it gives a maximum, minimum, or neither.

6.4 More than Two Variables

If there are n dependent variables in the original integral, there are n E-L equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f\left[x(u), y(u), x'(u), y'(u), u\right] du$$

with two dependent variables [$x(u)$ and $y(u)$] is stationary w.r.t variations of $x(u)$ and $y(u)$ iff these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{d}{du}\frac{\partial f}{\partial y'}$$

The indep. var in Lagrangian mechanics is the time t. The dep vars are the coordinates that specify the position of a system, and are usually denoted by $q_1, q_2, \ldots, q_n$. Because the coordinates $q_1, \ldots, q_n$ can take on so many guises, they are often referred to as generalized coordinates. It is often useful to think of the n generalized coordinates as defining a point in an n-dimensional configuration space, each of whose points labels a unique position of the system.

The ultimate goal in most problems in Lagrangian mechanics is to find how coordinates vary with time.

7

Lagrange's Equations

The Lagrange's formulation has two important advantages over the earlier Newtonian formulation. First, Lagrange's equations, unlike Newton's, take the same form in any coordinate system. Second, in treating constrained systems, such as a bead sliding on a wire, the Lagrangian approach eliminates the forces of constraint.

7.1 Lagrange's Equations for Unconstrained Motion

Consider a particle moving unconstrained in 3-d, subject to a conservative net force $\mathbf{F}(\mathbf{r})$. Kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and PE:

$$U = U(\mathbf{r}) = U(x, y, z)$$

The Lagrangian function, or just Lagrangian, is defined as

Lagrangian

$$\mathcal{L} = T - U$$

Remark:

> Notice first the Lagrangian is the KE *minus* the PE, not the same as the total energy. Notice also that I am using a script $\mathcal{L}$ for the Lagrangian and that $\mathcal{L}$ depends on the particle's position and its velocity; that is $\mathcal{L} = \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z})$.

Consider two derivatives

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x$$

Diff second eq w.r.t time, together with Newton's second law, $F_x = \dot{p}_x$, then

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \tag{7.1}$$

Then same for y and z. Thus we have shown Newton's second law implies the three Lagrange equations (in Cartesian coordinates so for):

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}, \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}, \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

Hamilton's Principle

The actual path which a particle follows between two points 1 and 2 in a given time interval, t_1 to t_2, is such that the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\; dt$$

is stationary when taken along the actual path.

So far, we have proved for a single particle that the following three statements are exactly equivalent:

1. A particle's path is determined by Newton's second law $\mathbf{F} = m\mathbf{a}$.
2. The path is determined by the three Lagrange equations, at least in Cartesian coordinates.
3. The path is determined by Hamilton's principle.

Remark:

There is one point about our derivation of Lagrange's equations that is worth keeping at the back of your mind. A crucial step in our proof was the observation that (7.1) was equivalent to Newton's second law, which in turn is true only if the original frame in which we wrote down $\mathcal{L} = T - U$ is inertial.

Example: **One Particle in Two Dimensions; Cartesian Coordinates**

The Lagrangian for a single particle in 2d is

$$\mathcal{L} = \mathcal{L}(x, y, \dot{x}, \dot{y}) = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)$$

The two Lagrange equations can be rewritten as follows:

$$\left.\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial x} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{x}} \quad \Longleftrightarrow F_x = m\ddot{x} \\[2ex] \dfrac{\partial \mathcal{L}}{\partial y} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{y}} \quad \Longleftrightarrow F_y = m\ddot{y} \end{array}\right\} \Longleftrightarrow \mathbf{F} = m\mathbf{a}$$

When we use generalized coordinates $q_1, \ldots, q_n$, we shall find that $\frac{\partial \mathcal{L}}{\partial q_i}$, although not necessarily a force component, plays a role similar to a force. Similarly, $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, although not necessarily a momentum component, acts very like a momentum. For this reason we shall call these derivatives the **generalized force** and **generalized momentum** respectively; that is

$$\frac{\partial \mathcal{L}}{\partial q_i} = (i\text{th component of generalized force })$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (i\text{th component of generalized momentum})$$

With each notations, each of the Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

takes the form

$$(\text{generalized force}) = (\text{rate of change of generalized momentum})$$

Example: **One Particle in Two Dimensions; Polar Coordinates**

The Lagrangian

$$\mathcal{L} = \mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r, \phi)$$

The r Equation

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

or

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

and $-\frac{\partial U}{\partial r}$ is just F_r, then

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

The ϕ Equation

is (by substitution)

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}(mr^2\dot{\phi})$$

Similarly ...

$$F_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi}$$

... Therefore, the ϕ equation states that

$$\Gamma = \frac{dL}{dt}$$

where Γ is the torque, L is the angular momentum.

Remark:

The last equation illustrates a wonderful feature of Lagrange's equations, that when we choose an appropriate set of generalized coordinates the corresponding Lagrange equations automatically appear in a corresponding natural form.

This example illustrates another feature of Lagrange's equations (in general): The ith component of the generalized force is $\frac{\partial \mathcal{L}}{\partial q_i}$. If this happens to be zero, then the Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

says simply that the ith component $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ of the generalized momentum is constant, or conserved.

7.2 Constrained Systems; an Example

Two familiar examples: bead threaded on a wire and rigid body, whose individual atoms can only move in such a way that the distance between any two atoms is fixed.

Consider the simple pendulum shown in Figure 7.1. massless rod, pivoted at O and free to swing without friction in xy plane.

We can express all quantities of interest in terms of ϕ. Then the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

If we choose the angle ϕ as our generalized coordinate, the Lagrange's equation reads

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$-mgl\sin\phi = \frac{d}{dt}\left(ml^2\dot{\phi}\right) = ml^2\ddot{\phi}$$

Referring to Figure 7.1, LHS is just the torque Γ exerted by gravity on the pendulum, while the term ml^2 is the pendulum's moment of inertia I. Since $\ddot{\phi}$ is the angular acceleration α, we see that Lagrange's equation for the simple pendulum simply reproduces the familiar result $\Gamma = I\alpha$.

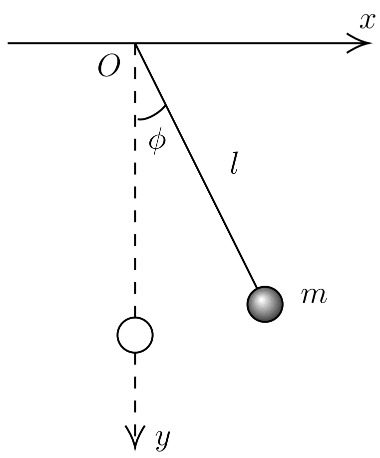


Figure 7.1: A simple pendulum.

The bob of mas m is constrained by the rod to remain distance l from O.

7.3 Constrained Systems in General

Generalized Coordinates

An arbitrary system of N particles, $\alpha = 1, \ldots, N$ with positions $\mathbf{r}_\alpha$. $q_1, \ldots, q_n$, and possibly with time t,

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, \ldots, q_n, t) \qquad [\alpha = 1, \ldots, N] \tag{7.2}$$

and conversely

$$q_i = q_i(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) \qquad [i = 1, \ldots, n]$$

In addition, we require that the number of the generalized coordinates (n) is the smallest number that allows the system to be parameterized in this way.

The double pendulum shown in Figure 7.2 has two bobs, both confined to a plane.

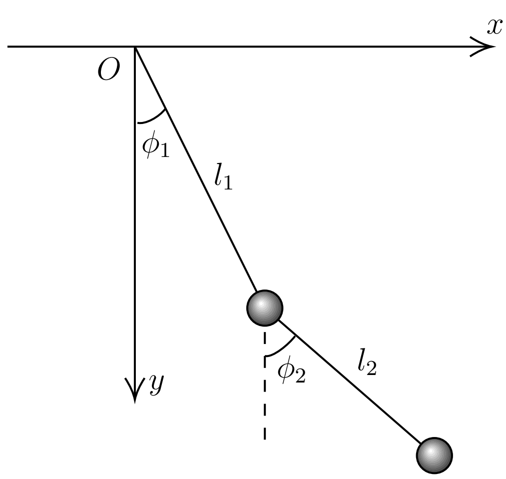


Figure 7.2: Double Pendulum

These four Cartesian coordinates can be expressed in terms of the two generalized coordinates ϕ_1 and ϕ_2.

$$\mathbf{r}_1 = (l_1 \sin\phi_1, l_1 \cos\phi_1) = \mathbf{r}_1(\phi_1)$$

and

$$\mathbf{r}_2 = (l_1 \sin\phi_1 + l_2 \sin\phi_2, l_1 \cos\phi_1 + l_2 \cos\phi_2) = \mathbf{r}_2(\phi_1, \phi_2)$$

We shall some times describe a set of coordinates $q_1, \ldots, q_n$ as **natural** if the relation (7.2) between the Cartesian coordinates $\mathbf{r}_\alpha$ and the generalized coordinates does not involve time t. Fortunately, as the name implies, there are many problems for which the most convenient choice of coordinates is also natural[1].

Degrees of Freedom

When the number of degrees of freedom of an N-particle system in 3 dimensions is less than $3N$, we say that the system is constrained.

In all of the examples so far, the number of freedom was equal to the number of generalized coordinates needed to describe the system's configuration. A system with this natural-seeming property is said to be **holonomic**[2].

holonomic

A holonomic system has n degrees of freedom and can be described by n generalized coordinates, $q_1, \ldots, q_n$.

Holonomic systems are easier to treat than nonholonomic.

For any holonomic system with generalized coordinates $q_1, \ldots, q_n$ and potential energy $U(q_1, \ldots, q_n, t)$, the evolution in time is determined by the n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad [i = 1, \ldots, n]$$

7.4 Proof of Lagrange's Equations with Constraints

We must recognize there are two kinds of forces on the particle:

1. the forces of constraint. In general, these forces are not necessarily conservative, but this doesn't matter.

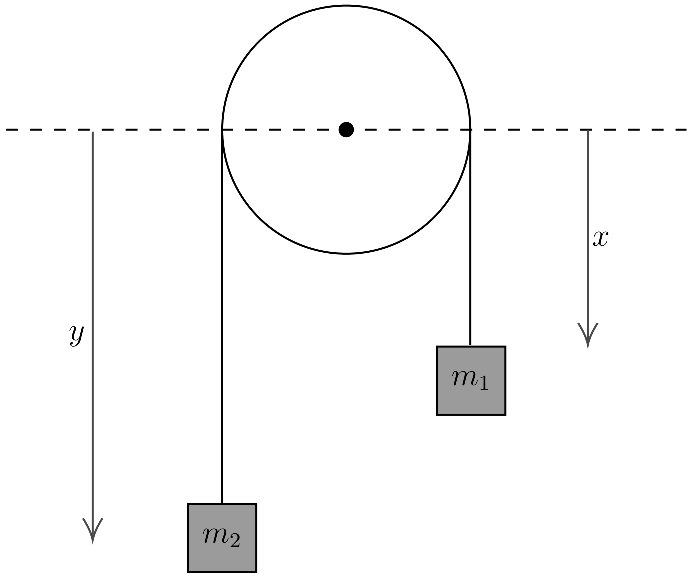
2. All the other "nonconstraint" forces on the particle, such as gravity.

[1] Natural are sometimes called scleronomous, and those are not natural, rheonomous.
[2] Many different defns can be found, not all of which are exactly equivalent

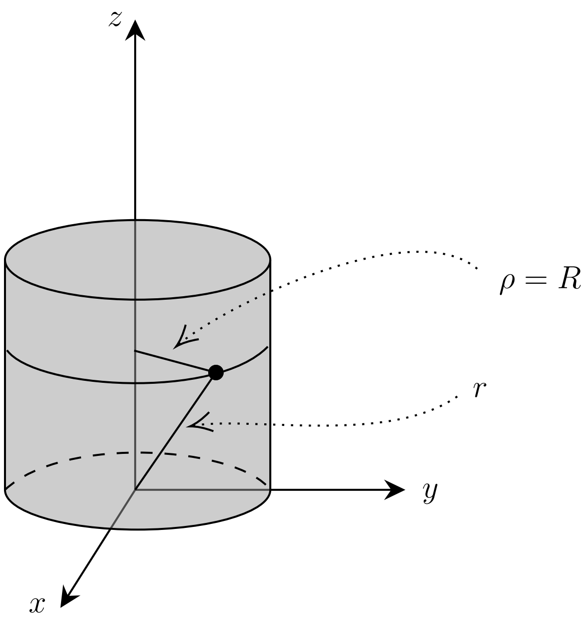
7.5 Examples of Lagrange's Equations

The Lagrangian method is so important that it certainly deserves more than just five examples. However, the crucial thing is that you work through several examples yourself; therefore the author has given plenty of problems at the end of the chapter.

Example: **Atwood's Machine**



Example: **A Particle Confined to Move on a Cylinder**



These two examples illustrate the steps:

1. Write down $\mathcal{L} = T - U$ using any convenient inertial reference frame.

2. Choose a convenient set of n generalized coordinates $q_1, \dots, q_n$ and find expressions for the original coordinates of step 1 in terms of your chosen generalized coordinates.

3. Rewrite $\mathcal{L}$ in terms of $q_1, \dots, q_n$ and $\dot{q}_1, \dots, \dot{q}_n$.

4. Write down the n Lagrange equations.

7.6 Generalized Momenta and Ignorable Coordinates

As previously mentioned, generalized forces and generalized momenta. Then the Lagrange equation can be rewritten as

$$F_i = \frac{d}{dt}p_i$$

When the Lagrangian is independent of a coordinate q_i, that coordinate is sometimes said to be ignorable or **cyclic**.

The result: "$\mathcal{L}$ is independent of a coordinate q_i" is equivalent to saying "$\mathcal{L}$ is unchanged, or invariant, when q_i varies (with all the other q_j held fixed)." This connection between invariance of $\mathcal{L}$ and certain conservation laws is the first of several similar results relating invariance under transformations to conservation laws. These results are known collectively as **Noether's theorem**. The author shall return to this theorem in Section 7.8.

7.7 Conclusion

Two great advantages:

- Unlike the Newtonian version, it works equally well in all coordinate systems and
- it can handle constrained system easily, avoiding any need to discuss the forces of generalized coordinates.

7.8 More about Conservation Laws*

**Advanced section. This is needed for Section 11.5 and Chapter 13.*

Conservation of Total Momentum

One of the most prominent features of an isolated system is that it is translationally invariant; that is if we transport all N particles bodily through the same displacement $\boldsymbol{\epsilon}$, nothing physically significant about the system should change.

We reach the conclusion that, provided the Lagrangian is unchanged by the translation $\mathbf{r}_i \to \mathbf{r}_i + \boldsymbol{\epsilon}$, the total momentum of the N-particle system is conserved. The connection between transnational invariance of $\mathcal{L}$ and conservation of total momentum is another example of Noether's Theorem.

Conservation of Energy

Detailed discussion in page 270 of the textbook.

Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

With this terminology, we can state the following important conclusion:

If the Lagrangian $\mathcal{L}$ does not dependent explicitly on time (that is, $\frac{\partial \mathcal{L}}{\partial t} = 0$), then the Hamiltonian $\mathcal{H}$ is conserved.

As we shall see in Chapter 13, the Hamiltonian $\mathcal{H}$ is the basis of the Hamiltonian formulation of mechanics, in just the same way that $\mathcal{L}$ is the basis of Lagrangian mechanics.

For the moment, the chief importance of our newly discovered Hamiltonian is that in many situations it is in fact just the total energy of the system. Specifically, we shall prove that, *provided the relation between the generalized coordinates and Cartesians is time-independent,*

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, \ldots, q_n),$$

the Hamiltonian $\mathcal{H}$ is just the total energy,

$$\mathcal{H} = T + U.$$

Proof:

See Page 271 of textbook. □

Remark:

- Thus we now see that time independence of the Lagrangian implies conservation of energy.
- Thus the result we have just proved is that invariance $\mathcal{L}$ under time translations is related to energy conservation, in much the same way that invariance of $\mathcal{L}$ under translations of space ($\mathbf{r} \to \mathbf{r} + \boldsymbol{\epsilon}$) is related to conservation of momentum.

Both results are manifestations of Noether's famous theorem.

7.9 Lagrange's Equations for Magnetic Forces*

**This section requires a knowledge of the scalar and vector potentials of electromag-*

netism. Although the ideas described here play an important role in the quantum-mechanical treatment of magnetic fields, they will not be used again in this book.

Defn and Nonuniqueness of Lagrangian

Lagrangian – a General Defn

For a given mechanical system with generalized coordinates $q = (q_1, \ldots, q_n)$, a Lagrangian $\mathcal{L}$ is a function $\mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ of the coordinates and velocities, such that the correct equations of motion for the system are the Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad [i = 1, \ldots, n]$$

Remark:

In other words, a Lagrangian is any function $\mathcal{L}$ for which Lagrange's equations are true for the system under consideration.

Old defn fits this new defn, but the new defn is much more general. In particular, new defn does not define a unique $\mathcal{L}$.

The crucial point is that ant function $\mathcal{L}$ which gives the right equation of motion has all of the features that we require of a Lagrangian and so is just as acceptable as any other such function $\mathcal{L}$.

Lagrangian for a Charge in a Magnetic Field

Note that this part won't be tested on the AMATH 271 exam according to the offering in Fall 2018.

A particle, mass m charge q, electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Then Lorentz force, with Newton's second law

$$m\ddot{\mathbf{r}} = q(E + \dot{r} \times B)$$

$$\vdots$$

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}$$

Generalized momentum = mechanical momentum + magnetic momentum

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8

Two-Body Central-Force Problems

8.1 The Problem

Two point particles, m_1 and m_2. Positions $\mathbf{r}_1$ and $\mathbf{r}_2$/ The only forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ of their mutual interaction, conservative and central. Thus potential energy

$$U(\mathbf{r}_1, \mathbf{r}_2) = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

For the electron and proton in hydrogen atom, the potential energy is the Coulomb PE of the two charges (e for the proton and $-e$ for the electron),

$$U(\mathbf{r}_1, \mathbf{r}_2) = -\frac{ke^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

In both examples, U only depends on magnitude $|\mathbf{r}_1 - \mathbf{r}_2|$, and we can write

$$U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|)$$

To take advantage of the equation above, we introduce a new variable

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

The U depends only on the magnitude r of relative position $\mathbf{r}$,

$$U = U(r)$$

Problem We want to find the possible motions of two bodies, whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(r)$$

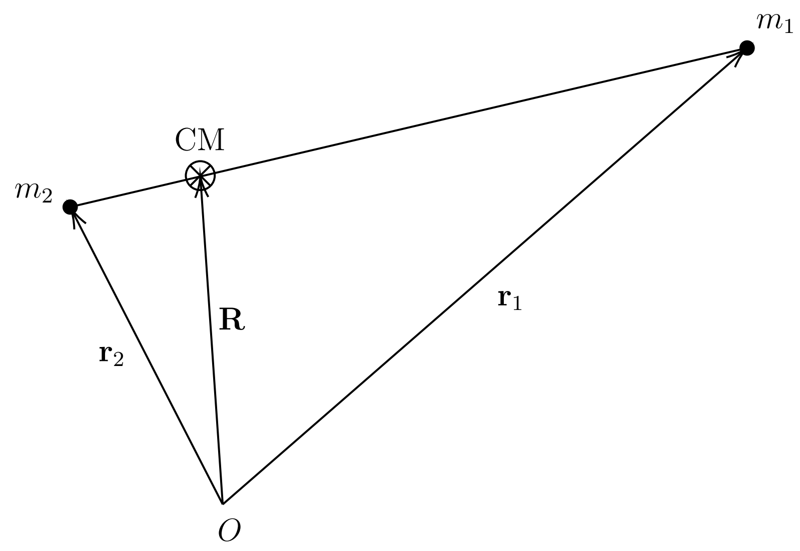


Figure 8.1: The center of mass of the two bodies

8.2 CM and Relative Coordinates; Reduced Mass

center of mass (CM)

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}$$

In Section 3.3, total momentum of the two bodies is the same as the total mass $M = m_1 + m_2$ were concentrated at CM and were following the CM as it moves:

$$\mathbf{P} = M\dot{\mathbf{R}}$$

It's extremely obvious in Figure 8.1 that

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r} \qquad \text{and} \qquad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

Thus the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}\left(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2\right) \\ &= \frac{1}{2}\left(m_1\left[\dot{\mathbf{R}} + \frac{m_2}{M}\mathbf{r}\right]^2 + m_2\left[\dot{\mathbf{R}} - \frac{m_1}{M}\mathbf{r}\right]^2\right) \\ &= \frac{1}{2}\left(M\dot{\mathbf{R}}^2 + \frac{m_1 m_2}{M}\mathbf{r}^2\right) \end{aligned}$$

The result simplifies further if we introduce the parameter

reduced mass

$$\mu = \frac{m_1 m_2}{M} \equiv \frac{m_1 m_2}{m_1 + m_2}$$

which has the dimensions of mass and is called the **reduced mass**. If $m_1 \ll m_2$, then μ is very close to m_1.

$$T = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2$$

Remark:

This remarkable result shows that the kinetic energy is the same as that of two "fictitious" particles, one of mass M moving with the speed of the CM, and the other mass μ (reduced mass) moving with the speed of the relative position $\mathbf{r}$. Even more significant is the corresponding result for the Lagrangian:

$$\mathcal{L} = T - U = \frac{1}{2} M \dot{\mathbf{R}}^2 + \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - U(r) \right) = \mathcal{L}_{cm} + \mathcal{L}_{rel}$$

8.3 The Equations of Motion

Because $\mathcal{L}$ is independent of $\mathbf{R}$, the $\mathbf{R}$ equation is especially simple,

$$M \ddot{\mathbf{R}} = 0 \qquad \text{or} \qquad \dot{\mathbf{R}} = \text{const}$$

We can explain this result in several ways: First, it is a direct consequence of conservation of total momentum. Alternatively, we can view it as reflecting that $\mathcal{L}$ is independent of $\mathbf{R}$.

The Lagrange eqn for $\mathbf{r}$ is a little less simple but equally beautiful:

$$\mu \ddot{\mathbf{r}} = -\nabla U(r)$$

The CM Reference Frame

Because $\dot{\mathbf{R}}$ =const, we can choose an inertial reference frame, the so-called **CM frame**, in which the CM is at rest and the total momentum is zero. In this frame $\dot{\mathbf{R}} = 0$ and $\mathcal{L}_{cm} = 0$. Thus in the CM frame

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2} \mu \dot{\mathbf{r}}^2 - U(r) \tag{8.1}$$

and the problem really is reduced to a one-body problem.

Conservation of Angular Momentum

In any frame,

$$\begin{aligned}\mathbf{L} &= \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 \\ &= m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2\end{aligned}$$

In CM frame, set $\mathbf{R} = 0$:

$$\mathbf{r}_1 = \frac{m_2}{M}\mathbf{r} \qquad \text{and} \mathbf{r}_2 = -\frac{m_1}{M}\mathbf{r}$$

Then

$$\begin{aligned}\mathbf{L} &= \frac{m_1 m_2}{M^2}\left(m_2\mathbf{r} \times \dot{\mathbf{r}} + m_1\mathbf{r} \times \dot{\mathbf{r}}\right) \\ &= \mathbf{r} \times \mu\dot{\mathbf{r}}\end{aligned}$$

The most remarkable thing about this result is that the total angular momentum in the CM frame is exactly the same as the angular momentum of a single particle with mass μ and position $\mathbf{r}$.

The Two Equations of Motion

The obvious choice is to use r and ϕ, then Lagrangian (8.1) is

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

Since the Lagrangian is independent of ϕ, the coordinate ϕ is ignorable, and the Lagrange equation corresponding to ϕ is just

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \implies 0 = \frac{d}{dt}(\mu r^2\dot{\phi}) \implies \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \mu r^2\dot{\phi} = \text{const} = \ell \qquad [\phi \text{ equation}]$$

Since $\mu r^2\dot{\phi}$ is the angular momentum ℓ, the ϕ equation is just a statement of conservation of angular momentum.

The Lagrange equation corresponding to r (often called the radial equation) is

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}}$$

or

$$\mu\ddot{r} = \underbrace{-\frac{\mathrm{d}U}{\mathrm{d}r}}_{\text{Gravity}} + \underbrace{\mu r\dot{\phi}^2}_{\text{Centrifugal}}$$

8.4 The Equivalent One-Dimensional Problem

Our main use for the ϕ equation is to solve it for $\dot{\phi}$,

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

which will let us eliminate $\dot{\phi}$ from the radial equation in favor of the constant ℓ. Then the radial eqn can be rewritten as

$$\mu\ddot{r} = -\frac{dU}{dr} + \mu r\ddot{\phi}^2 = -\frac{dU}{dr} + F_{cf}$$

where F_{cf} is the centrifugal force. This equation has the form of Newton second law for a particle in one dimension, subject to the actual force $-\frac{dU}{dr}$ plus the centrifugal force.

$$F_{cf} = \mu r\dot{\phi}^2 = \frac{\ell^2}{\mu r^3}$$

Even better, we can now express the centrifugal force in terms of a centrifugal potential energy,

$$F_{cf} = -\frac{d}{dr}\left(\frac{\ell^2}{2\mu r^2}\right) = -\frac{dU_{cf}}{dr}$$

Now we can rewrite the radial equation in terms of U_{cf} as

$$\mu\ddot{r} = -\frac{d}{dr}[U(r) + U_{ef}(r)] = -\frac{d}{dr}U_{eff}(r)$$

where the **effective potential energy** $U_{eff}(r)$ is the sum of the actual potential energy $U(r)$ and the centrifugal $U_{cf}(r)$.

Example: **Effective Potential Energy for a Comet**

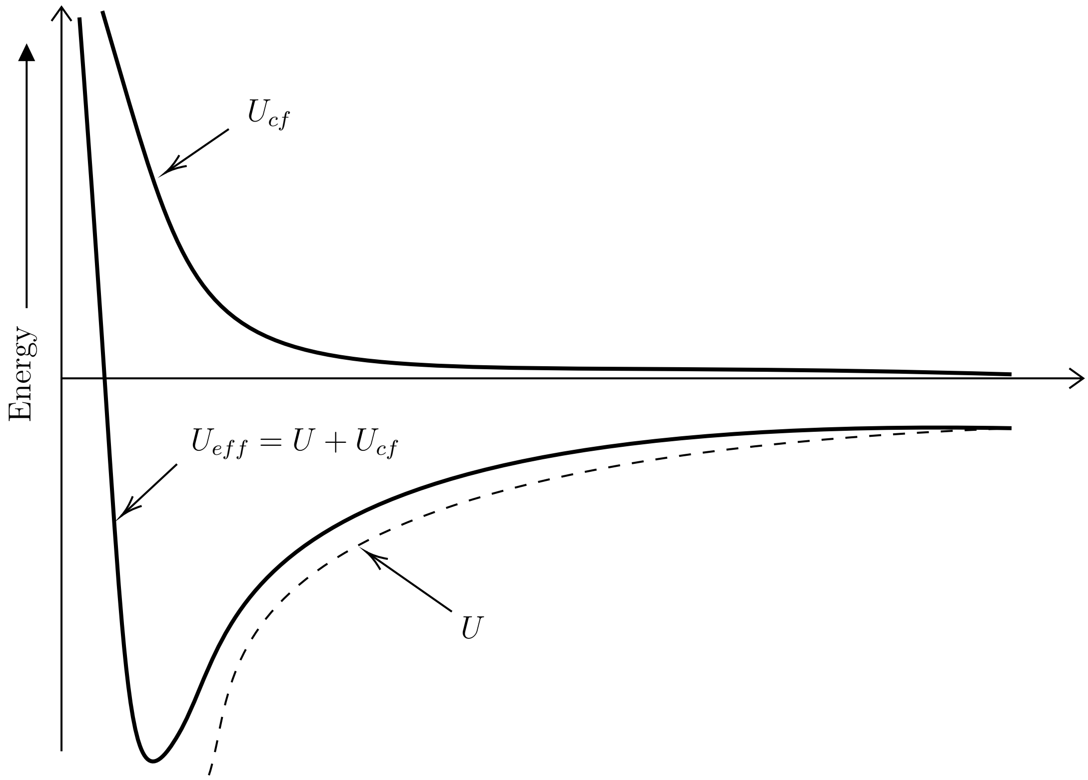
Since planetary motion was first described mathematically by the German astronomer Johannes Kepler, 1571-1630, this problem of the motion of a planet or comet around the sun (or any two bodies interacting via an inverse-square force) is often called the *Kepler problem*.

The actual gravitational potential energy of the comet is given by the well-known formula

$$U(r) = -\frac{Gm_1m_2}{r}$$

The total effective potential energy is

$$U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$



Conservation of Energy

To find the details of the orbit, we must look more closely at the radial equation. If we multiply both sides by $\dot{r}$,

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}U_{eff}(r) \implies \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = \text{const.}$$

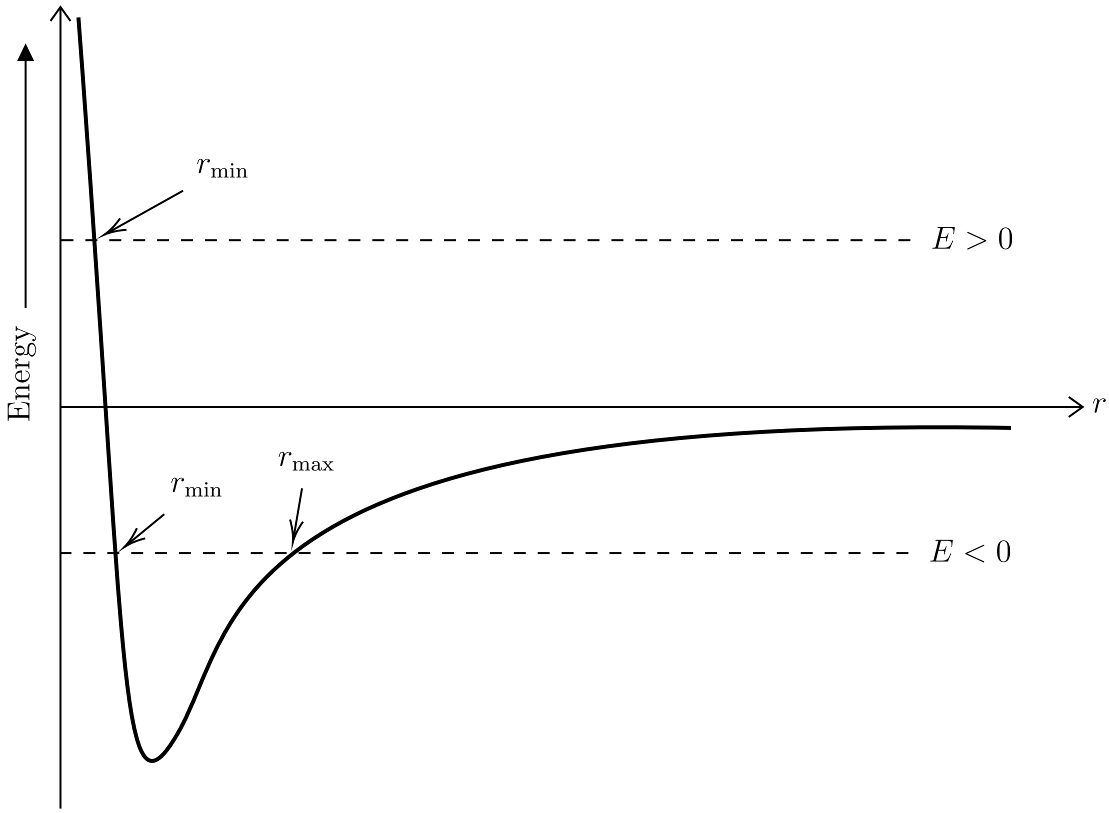
In fact, this result is just conservation of energy. If we replace U_{eff} and ℓ, we see that

$$\begin{aligned} \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r) \\ &= E \end{aligned} \tag{8.2}$$

Example: **Energy Considerations for a Comet or Planet**

Examine again the previous example and by considering its total energy E, find the equation determines the max/min dist of the comet from the sun.

In the energy equation (8.2) the term $\frac{1}{2}\mu\dot{r}^2$ on the left is always greater than or equal to zero. Therefore, the comet's motion is confined to those regions where $E \geq U_{eff}$. Two cases:



1. $E > 0$. A comet with this energy can move anywhere above the curve of $U_{eff}(r)$, but nowhere that the line is below the curve.

$$U_{eff}(r_{\min}) = E$$

   If the comet is initially moving toward the sun, then it will continue until it reaches $r_{\min}$, where $\dot{r} = 0$ instantaneously. It then moves outward, and since there are no other points at which $\dot{r}$ can vanish, it eventually moves off to infinity, and the orbit is **unbounded**.

2. $E < 0$. Trapped between these two values of r. **bounded orbit**.

3. $E = \min U_{eff}(r)$, the two turning points coalesce, and the comet is trapped at a fixed radius and moves in a circular orbit.

8.5 The Equation of the Orbit

The radial equation determines r as a function of t, but for many purposes we would like to know r as a function of ϕ.

First write the radial eqn in terms of forces:

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}$$

where $F(r)$ is the actual central force, $F = -dU/dr$, and the second term is the centrifugal force.

Two tricks:

1. rewrite the equation in terms of ϕ and make the substitution $u = \frac{1}{r}$.
2. rewrite the diff operator d/dt in terms of $d/d\phi$ using the chain rule.

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$$

Sub back into the radial eqn we find

$$-\frac{\ell^2 u^2}{\mu}\frac{d^2 u}{d\phi^2} = F + \frac{\ell^2 u^3}{\mu}$$

or

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} F \tag{8.3}$$

8.6 The Kepler Orbits

Let us now return to the Kepler problem, the problem of finding the possible orbits of a comet or any other object to an inverse-square force. To include both cases (gravitational force, Coulomb force),

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2$$

where γ is the "force constant".

Inserting it into (8.3), we find $u(\phi)$ must satisfy

$$u''(\phi) = -u(\phi) + \frac{\gamma\mu}{\ell^2}$$

$$\vdots$$

The general solution for $u(\phi)$

$$u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi = \frac{\gamma\mu}{\ell^2}(1 + \epsilon\cos\phi)$$

I shall introduce new length

$$c = \frac{\ell^2}{\gamma\mu}$$

in terms of which our solution becomes

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi} \tag{8.4}$$

The Bounded Orbits

A glance at 8.4 shows this behavior is very different according as $\epsilon < 1$ or $\epsilon \geq 1$.

With $\epsilon < 1$, $r(\phi)$ oscillates between $r_{\text{min}} = \frac{c}{1+\epsilon}$ and $r_{\text{max}} = \frac{c}{1-\epsilon}$.

$r = r_{\text{min}}$: perihelion 近日点, when $\phi = 0$ and $r = r_{\text{max}}$: aphelion 远日点, when $\phi = \pi$.

Rewrite (8.4) in Cartesian coordinates:

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$a = \frac{c}{1-\epsilon^2}, \qquad b = \frac{c}{\sqrt{1-\epsilon^2}}, \qquad \text{and} \qquad d = a\epsilon$$

Observe that

$$\frac{b}{a} = \sqrt{1-\epsilon^2}$$

is the defn of eccentricity of the ellipse. That is, ϵ is the eccentricity.

Thus the position of the sun is one of the ellipse's two focuses. and we have now proved **Kepler's first law**, that the planets follow orbits that are ellipses with the sun at one focus.

The Orbital Period; Kepler's Third Law

Together with Kepler's second law,

$$\frac{dA}{dt} = \frac{\ell}{2\mu}$$

$$\vdots$$

We find that

$$\tau^2 = \frac{4\pi^2}{GM_{sun}} a^3 \tag{8.5}$$

This is **Kepler's third law**:for all bodies orbiting the sun, the square of period is proportional to the cube of the semimajor axis.

Relation between Energy and Eccentricity

$$\left.\begin{aligned} E &= U_{eff}(r_{\text{min}}) \\ r_{\text{min}} &= \frac{c}{1+\epsilon} \end{aligned}\right\} \implies E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1)$$

which is equally valid for bounded and unbounded orbits.

8.7 The Unbounded Kepler Orbits

Unbounded: $\epsilon \geq 1$ and $E \geq 0$.

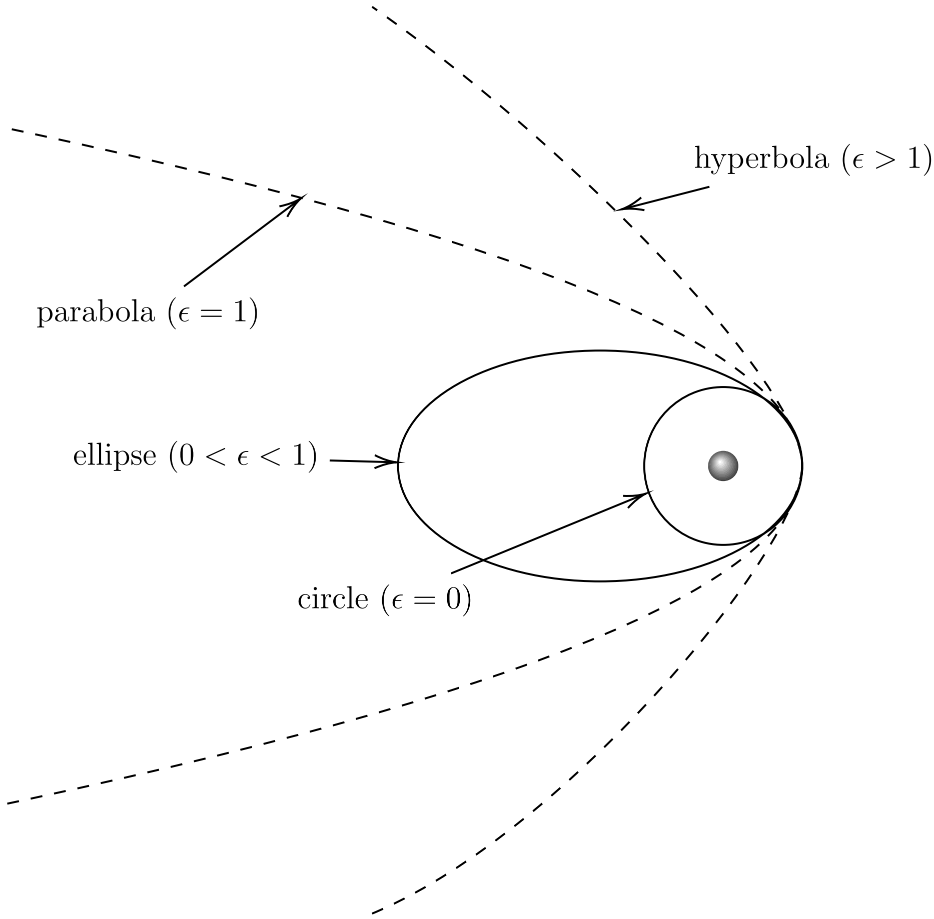


Figure 8.2: Four different Kepler orbits

eccentricity	energy	orbit
0	< 0	circle
$(0, 1)$	< 0	ellipse
1	0	parabola
> 1	> 0	hyperbola

8.8 Changes of Orbit

In this final section, I shall discuss how a satellite can change from one orbit to another.

perigee and apogee

$$r_1(\phi_0) = r_2(\phi_0) \implies \frac{c_1}{1 + \epsilon_1 \cos(\phi_o - \delta_1)} = \frac{c_2}{1 + \epsilon_2 \cos(\phi_o - \delta_2)}$$

A Tangential Thrust at Perigee

$v_2 = \lambda v_1$, where λ is the **thrust factor**. Unlikely, it will be negative.

$$\ell_2 = \lambda \ell_1 \implies c_2 = \lambda^2 c_1 \implies \epsilon_2 = \lambda^2 \epsilon_1 + (\lambda^2 - 1)$$

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9

Mechanics in Noninertial Frames

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10 Rotational Motion of Rigid Bodies

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11

Coupled Oscillators and Normal Modes

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12 Nonlinear Mechanics and Chaos

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13

Hamiltonian Mechanics

13.1 The Basic Variables

$$\mathcal{L} = \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = T - U$$

The n coordinates $(q_1, \ldots, q_n)$ define a point in an n-dimensional configuration space. The 2n coordinates $(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ define a point in state space, and specify a set of ICs that determine a unique soln of the n 2nd order DEs of motions, Lagrange's eqns.

Recall we also define generalized momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

and it's also called the canonical momentum or the momentum conjugate to q_i.

Hamiltonian function or just Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

In Section 7.8, we proved that , provided generalized coordinates are "natural" (that is, the relation between the q's and the underlying Cartesian coordinates is time independent), $\mathcal{H}$ is just the total energy of the system and is, therefore, familiar and easy to visualize.

Second important difference between Lagrangian and Hamiltonian. In Hamiltonian. we shall use

$$(q_1, \ldots, q_n, p_1, \ldots, p_n)$$

which is usually called phase space.

Like Lagrange's approach, Hamilton's is best suited to systems that are subject to no frictional forces.

13.2 Hamilton's Equations for 1d Systems

$$\mathcal{L} = \mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

General result in Section 7.8 that the Lagrangian for a conservative system with "natural" coordinates (1d here) has the general form:

$$\mathcal{L} = \mathcal{L}(q, \dot{q}) = T - U = \frac{1}{2}A(q)\dot{q}^2 - U(q) \tag{13.1}$$

Then Hamiltonian reduces to

$$\mathcal{H} = p\dot{q} - \mathcal{L} \tag{13.2}$$

Given the form (13.1) of $\mathcal{L}$, we can calculate the generalized momentum p as

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = A(q)\dot{q} \tag{13.3}$$

so that $p\dot{q} = A(q)\dot{q}^2 = 2T$. Sub into (13.2), we find that

$$\mathcal{H} = p\dot{q} - \mathcal{L} = T + U$$

$\mathcal{H}$ for the "natural" system considered here is precisely the total energy.

The next step is to set up the Hamiltonian formalism. We can solve (13.3) for $\dot{q}$ in terms of q and p:

$$\dot{q} = \frac{p}{A(q)} = \dot{q}(q, p)$$

In all its horrible detail, (13.2) becomes

$$\mathcal{H}(q, p) = p\dot{q}(q, p) - \mathcal{L}(q, \dot{q}(q, p)) \tag{13.4}$$

Our final step is to get Hamilton's equation of motion (recognize $\frac{\partial \mathcal{L}}{\partial \dot{q}} = p$)

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial q} &= p\frac{\partial \dot{q}}{\partial q} - \left[\frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}\right] \\
&= -\frac{\partial \mathcal{L}}{\partial q} \\
&= -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \\
&= -\frac{d}{dt}p \\
&= -\dot{p}
\end{aligned}$$

and

$$\frac{\partial \mathcal{H}}{\partial p} = \left[\dot{q} + p\frac{\partial \dot{q}}{\partial p}\right] - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p} = \dot{q}$$

Then we have Hamilton's eqns for one-dimensional system:

Hamilton's Equations

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \qquad \text{and} \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{13.5}$$

Here we have two first-order eqns.

Example: **Atwood's Machine**

$$\mathcal{L} = T - U$$

where

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 \qquad \text{and} \qquad U = -(m_1 - m_2)gx$$

then ...

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m_1 + m_2} \qquad \text{and} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = (m_1 - m_2)g$$

Then

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

Our first task is always to write down the $\mathcal{H}$.

Once this is done, one can just turn the handle and crank out Hamilton's eqns.

13.3 Hamilton's Eqns in Several Dimensions

Here we shall use the abbreviation introduced in Section 11.5:

$$\mathbf{q} = (q_1, \ldots, q_n) \qquad \dot{\mathbf{q}} = (\dot{q}_1, \ldots, \dot{q}_n) \qquad \mathbf{p} = (p_1, \ldots, p_n)$$

With necessary assumptions to guarantee the standard Lagrangian formalism applies, let's derive from it the Hamiltonian one. Thus Lagrangian

$$\mathcal{L} = \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T - U$$

$$\vdots$$

Hamilton's equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \qquad [i = 1, \ldots, n]$$

$$\frac{d\mathcal{H}}{dt} = \sum_{i=1}^{n} \left[\frac{\partial \mathcal{H}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i \right] + \frac{\partial \mathcal{H}}{\partial t} \implies \frac{d\mathcal{H}}{t} = \frac{\partial \mathcal{H}}{\partial t}$$

Remark:

Note the difference between two derivatives. d-one (sometimes called the total derivative), is the actual rate of change of $\mathcal{H}$ as the motion proceeds, with all the coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ changing as t advances. ∂-one is the rate of change of $\mathcal{H}$ if we vary its last argument t holding all the other argument fixed. In particular, if $\mathcal{H}$ does not depend explicitly on t, this partial derivative will be zero.

This simple result indicates that $\mathcal{H}$ varies with time only to the extent that it is explicitly time dependent. In particular, if $\mathcal{H}$ does not depend explicitly on t, then $\mathcal{H}$ is conserved.

6 steps to Find Hamiltonian

1. Find the generalized coordinates q_i
2. Write T and U in terms of q's and $\dot{q}$'s
3. Find generalized momentum $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$
4. Write $\dot{q}_i$ in terms of p_i and q_i
5. Find $\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$ ($\mathcal{H} = T + U = E$ for natural coordinates)
6. Solve Hamilton's equations and find the equation of motion.

13.4 Ignorable Coordinates

If $\mathcal{L}$ happens to be independent of a coordinate q_i, then the corresponding generalized momentum p_i is constant. When this happens, we say that the coordinate q_i is **ignorable**.

13.5 Lagrange's Equations vs Hamilton's Equations

First we can rewrite the first n of Equations (13.5) as

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = f_i(\mathbf{q}, \mathbf{p})$$

Combine these n eqns

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \mathbf{p}) \tag{13.6}$$

Similarly,

$$\dot{\mathbf{p}} = \mathbf{g}(\mathbf{q}, \mathbf{p}) \tag{13.7}$$

Finally, we can introduce a $2n$-dimensional vector

$$\mathbf{z} = (\mathbf{q}, \mathbf{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$$

phase-space vector or phase point $\mathbf{z}$. Each value of $\mathbf{z}$ labels a unique point in phase space and identifies a unique set of initial conditions for our systems. Combine (13.6) and (13.7)

$$\dot{\mathbf{z}} = \mathbf{h}(\mathbf{z})$$

We can consider changes of coordinates of the form

$$\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{p}) \quad \text{and} \quad \mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{p})$$

that is, coordinate changes in which both the q's and the p's are intermingled. If the eqns above satisfy certain conditions, this change of coordinates is called a **canonical transformation**, and it turns out that Hamilton's eqns are invariant under these canonical transformations. Further discussion is out of scope, but note that there is no corresponding transformation in Lagrangian mechanics.

13.6 Phase-Space Orbits

Any point $\mathbf{z}_o$ defines a possible initial condition (at any chosen time t_0), and Hamilton's equations define a unique phase-space orbit or trajectory which starts from $\mathbf{z}_0$ at t_0 and which the system follows as time progresses.

Important Property: No two different phase-space orbits can pass through the same point in phase space; that is, no two phases-space orbits can cross another. One can imagine two orbits passing through the same point zo in phase space. However, Hamilton's equations guarantee that for any given point $\mathbf{z}_0$, there is a unique orbit passing through $\mathbf{z}_0$, so the two orbits must in fact be the same.

It has important consequences in, for example, the analysis of chaotic motion of Hamiltonian systems.

13.7 Liouville's Theorem

Theorem 13.1: Liuville's Theorem

If we imagine a large number of identical systems launched at the same time with slightly different initial conditions, the phase-space points that represent the systems can be seen as forming a fluid. Liouville's theorem states that the density of this fluid is constant in time (or, equivalently, that the volume occupied by any group of points is constant.

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14

Collision Theory

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15 Special Relativity

15.1 Relativity

The theory of relativity is the study of the consequences of the relativity of measurements. Einstein's relativity is really two theories.

- Special relativity is "special" in that it focuses primarily on unaccelerated frames of reference.
- General relativity is "general" in that it includes accelerated reference frames.

15.2 Galilean Relativity

In classical physics, Newton's laws are invariant as we transfer our attention from one inertial frame to another. The classical transformation from one frame to a second, moving at constant velocity to the first, is called the **Galilean transformation**.

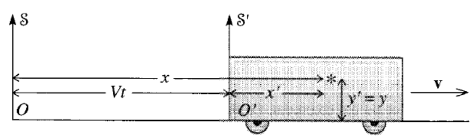


Figure 15.1 The frame $\mathcal{S}$ is fixed to the ground, while $\mathcal{S}'$ is fixed in a railroad car traveling with constant velocity **V** in the x direction. The two origins coincide, $O = O'$, at time $t = t' = 0$. The star indicates an event, such as a small explosion.

Figure 15.1: excerpt from book

Galilean transformation

Consider two frames $\mathcal{S}$ and $\mathcal{S}'$ that are oriented the same way. Suppose further that the velocity $\mathbf{V}$ of $\mathcal{S}'$ relative to $\mathcal{S}$ is along the x axis. The configuration is illustrated in Figure 15.1.

By the classical assumption concerning time, $t' = t$, so the required relations are

$$\left.\begin{aligned} x' &= x - Vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \tag{15.1}$$

These four equations are called the **Galilean transformation**. They are the mathematical expression of the classical ideas about space and time.

(15.1) relates the coordinates measured in two frames arranged with corresponding axes parallel and with relative velocity along the x axis, as shown in Figure 15.1 – an arrangement we can call the **standard configuration**. This is not general. For example, if relative velocity $\mathbf{V}$ is in an arbitrary direction, (15.1) can be rewritten compactly as

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t \qquad \text{and} \qquad t' = t \tag{15.2}$$

This is still not the most general since we could rotate the axes. However, (15.2) is enough for our present purposes. If we differentiate (15.2)

$$\mathbf{v}' = \mathbf{v} - \mathbf{V}$$

This is the **classical velocity-addition formula**.

Galilean Invariance of Newton's Laws

Prove 3 laws hold.

Galilean Relativity and the Speed of Light

While Newton's laws are invariant under the Galilean transformation, the same is not true of the laws of electromagnetism. Whether we write them in their compact form as Maxwell's four equations, or in their original form (as Coulomb's law, Faraday's law, and so on), they can be true in one inertial frame, but if they are, and *if the Galilean transformation were the correct relation between different inertial frames*, then they could not be true in any other inertial frame.

If the Galilean transformation were the correct transformation between inertial frames, then although Newton's laws would hold in all inertial frames, there could only be one frame in which Maxwell's equations hold. This supposed unique frame, in which light would travel at the same speed in all directions, is sometimes called the ether frame.[1]

The Michelson-Morley Experiment

interferometer.

With hindsight, it is easy to draw the right conclusion: Contrary to all expectations, the speed of light is the same in all directions relative to an earth-based frame, even though the earth has different velocities at different times of year. In other words, it is not true that there is only one frame in which light has the same speed in all directions.

None of these alternative theories was able to explain all of the observed facts (at least, not in a reasonable and economical way), and today almost all physicists accept that there is no unique ether frame and that the speed of light is a universal constant, with the same value in all directions in all inertial frames.

15.3 The Postulates of Special Relativity

The special theory of relativity is based on the acceptance of the universality of the speed of light, as suggested by the Michelson-Morley experiment.

Definition of an Inertial Frame

An inertial frame is any reference frame (that is, a system of coordinates x, y, z and time t) in which all the laws of physics hold in their usual form.

Notice "all the laws of physics" have not been specified. Following Einstein, we shall use the postulates of relativity to help us decide what the laws of physics could be. (As always, the ultimate test will be whether they agree with experiment.) The big difference between the inertial frames of relativity and those of classical mechanics is the mathematical relation between different frames. In relativity, we shall find that the classical Galilean transformation must be replaced by the so-called Lorentz transformation.

First Postulate of Relativity

If $\mathcal{S}$ is an inertial frame and if a second frame $\mathcal{S}'$ moves with constant velocity relative to $\mathcal{S}$, then $\mathcal{S}'$ is also an inertial frame.

[1]The origin of the name is this: It was assumed that light must propagate through a medium, in much the same way that sound travels through the air. Since no one had ever detected this medium and since light could travel through seemingly empty space, the medium clearly had most unusual properties, and was named "ether" after the Greek for the stuff of the heavens. The "ether frame" was the frame in which the supposed ether was at rest.

Remark:

Another popular statement of the first postulate is that "there is no such thing as absolute motion."

Yet another statement of the first postulate is that among all the inertial frames, there is no preferred frame. The laws of physics single out no one frame as being in any way more special than any other.

Second Postulate of Relativity

The speed of light (in vacuum) has the same value c in every direction in all inertial frames.

Remark:

This is, of course, the Michelson-Morley result.

15.4 The Relativity of Time; Time Dilation

Measurement of Time in a Single Frame

In what follows, I shall assume that any inertial frame $\mathcal{S}$ comes with a set of rectangular axes $Oxyz$ and a team of helpers stationed at rest throughout $\mathcal{S}$ and equipped with synchronized clocks. This allows us to assign a position (x, y, z) and a time t to any event, as observed in the frame $\mathcal{S}$.

Time Dilation

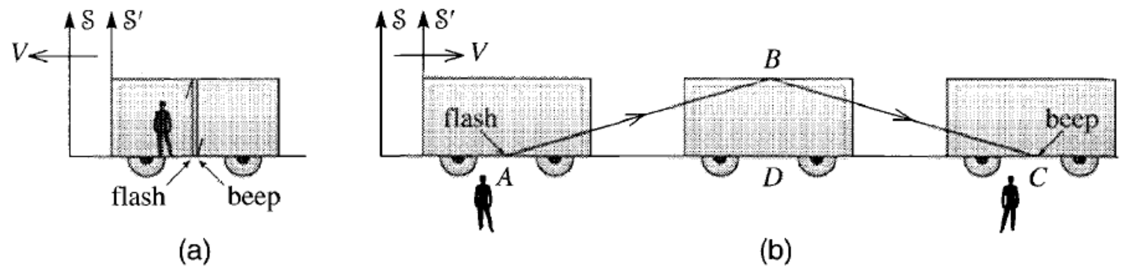
Let us now compare measurements of time made by observers in two different inertial frames. $\mathcal{S}$ anchored to the ground and $\mathcal{S}'$ traveling with a train in the x direction at speed V relative to $\mathcal{S}$. We now examine a **thought experiment**[2] in which an observer on the train sets off a flashbulb on the floor of the train. The light travels to the roof, where it is reflected back and returns to its starting point, where it strikes a photocell and causes an audible "beep." We wish to compare the times, Δt and $\Delta t'$, as measured in the two frames, between the flash as the light leaves the floor and the beep as it returns. See Figure 15.2.

$$\Delta t' = \frac{2h}{c} \tag{15.3}$$

and

$$(c\Delta t/2)^2 = h^2 + (V\Delta t/2)^2$$

[2] or gedanken experiment, from the German



~~Figure 15.1~~ 2 **(a)** The thought experiment as seen in frame $\mathcal{S}'$. The light travels straight up and down again, and the flash and beep occur at the same place. **(b)** As seen in $\mathcal{S}$, the flash and beep are separated by a distance $V\,\Delta t$. Notice that in $\mathcal{S}$ two observers are needed to time the two events, since they occur in different places.

Figure 15.2: another excerpt from textbook

which gives us

$$\Delta t = \frac{2h}{\sqrt{c^2 - V^2}} = \frac{2h}{c}\frac{1}{\sqrt{1-\beta^2}} \tag{15.4}$$

where I have introduced the useful abbreviation

$$\beta = \frac{V}{c}$$

which is just the speed V measured in units of c.

Combining (15.3) and (15.4),

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}}$$

Remark:

For $0 \le V < c$, $\Delta t \ge \Delta t'$

If $V = c$, then denominator $= 0$. If $V > c$, then imaginary value for $\Delta t'$. These results suggest that V must always be less than c, and this gives us one of the most profound results of relativity:

The relative speed of two inertial frames can never equal or exceed c.

Usually we have this notation

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\Delta t = \gamma \Delta t' \ge \Delta t'$$

To emphasize this asymmetry, the time $\Delta t'$ is often renamed Δt_0, then

$$\Delta t = \gamma \Delta t_0 \geq \Delta t_0$$

Δt_0 is often called the proper time. The effect is called time dilation.

Evidence for Time Dilation

B. Rossi and D. B. Hall, half-life.

15.5 Length Contraction

$$l = V\Delta t$$

$$l' = V\Delta t'$$

Then

$$l' = \gamma l$$

or

$$l = \frac{l'}{\gamma} \leq l'$$

Also, it is common to rewrite it as

$$l = \frac{l_0}{\gamma} \leq l_0$$

where l_0 denotes the length of an object measured in the object's **rest frame**, while l is the length in any frame. l_0, proper length. and **length contraction**.

Lengths Perpendicular to the Relative Velocity

We conclude that lengths perpendicular to the relative motion are unchanged. The length-contraction formula applies only to lengths parallel to the relative velocity.

15.6 The Lorentz Transformation

Since lengths perpendicular to relative velocity are the same in both frames, then

$$y' = y \qquad \text{and} \qquad z' = z$$

exactly as with the Galilean transformation. and by the length contraction formula (proper length x')

$$x - Vt = \frac{x'}{\gamma}$$

or

$$x' = \gamma(x - Vt)$$

and with a simple trick

$$t' = \gamma\left(t - \frac{Vx}{c^2}\right)$$

Collecting all equations we have

The Lorentz Transformation

$$\left.\begin{array}{l} x' = y(x - Vt) \\ y' = y \\ z' = z \\ t' = y\left(t - Vx/c^2\right) \end{array}\right\}$$

Remark:

These four equations are called the Lorentz transformation or the Lorentz—Einstein transformation.

and inverse Lorentz transformation

$$\left.\begin{array}{l} x' = y(x + Vt) \\ y' = y \\ z' = z \\ t' = y\left(t + Vx/c^2\right) \end{array}\right\}$$

Example: **A Relativistic Snake**

Details on page 613-615 of the textbook.

The resolution of this paradox, and many similar paradoxes, is that two events that are simultaneous in one frame are not necessarily simultaneous in a different frame – an effect sometimes called the **relativity of simultaneity**. As soon as we recognize that the two cleavers fall at different times in the snake's frame, there is no longer any problem understanding how they can both contrive to miss the snake.

15.7 The Relativistic Velocity-Addition Formula

Consider a particle moving with position $\mathbf{r}(t)$ or $\mathbf{r}'(t')$, as seen in $\mathcal{S}$ or $\mathcal{S}'$. The defn of the velocity $\mathbf{v}$ is the derivative

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

Now we can write down the Lorentz transformation and take the differences, we find

$$dx' = \gamma(dx - V dt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma\left(dt - V dx/c^2\right)$$

Then ...

relativistic velocity-addition formulas

$$\begin{cases} v'_x = \dfrac{v_x - V}{1 - v_x V/c^2} \\ \\ v'_y = \dfrac{v_y}{1 - v_y V/c^2} \\ \\ v'_z = \dfrac{v_z}{1 - v_z V/c^2} \end{cases}$$

15.8 Four-Dimensional Space - Time; Four-Vectors

Rotations of Ordinary Three-Dimensional Space

$$\mathbf{q} = \sum_{i=1}^{3} q_i \mathbf{e}_i$$

where

$$q_i = \mathbf{e}_i \cdot \mathbf{q}$$

To conform this notation, I shall from now on, rename the position vector $\mathbf{r} = (x, y, z)$ as $\mathbf{x} = (x_1, x_2, x_3)$.

Now consider a new set of unit vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. The components q'_i are easily found

$$q'_i = \mathbf{e}'_i \cdot \mathbf{q} = \mathbf{e}'_i \cdot \sum_{j=1}^{3} q_j \mathbf{e}_j = \sum_{j=1}^{3} \left(\mathbf{e}'_i \cdot \mathbf{e}_j\right) q_j$$

We can express the eqn above more compactly. Let $\mathbf{R}$ be a 3×3 matrix with elements

$$R_{ij} = e'_i \cdot e'_j$$

and $\mathbf{q}, \mathbf{q}'$ denote the coordinates

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad \text{and} \quad \mathbf{q}' = \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix}$$

With these notations, we have

$$\mathbf{q}' = \mathbf{R}\mathbf{q}$$

$\mathbf{R}$ is a 3×3 rotation matrix.

Lorentz Transformation as "Rotations" of Space-Time

Introduce a fourth coordinate

$$x_4 = ct$$

where c guarantees that x_4 has the same dimensions as x_1, x_2 and x_3. Recall $\beta = V/c$, then we can rewrite the Lorentz transformation as

$$\left.\begin{aligned} x_1' &= \gamma x_1 - \gamma\beta x_4 \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= -\gamma\beta x_1 + \gamma x_4 \end{aligned}\right\}$$

Four-Vectors

The four numbers x_1, x_2, x_3 and $x_4 = ct$ constitute a vector in four-dimensional space-time. Such vectors are called **four-vectors**. We will be using ordinary italic letters for four vectors:

$$q = (q_1, q_2, q_3, q_4) = (\underbrace{\mathbf{q}}_{\text{spatial component}}, \underbrace{q_4}_{\text{time component}})$$

With this notation, the Lorentz transformation can be written in matrix form as

$$x' = \Lambda x$$

where Λ is the 4×4 matrix

$$\Lambda = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \quad \text{[standard boost]}$$

This is not the most general Lorentz transformation. For many purposes, this **standard transformation** is the only one we need to consider.

Any Lorentz transformation which leaves corresponding axes parallel is called a **pure boost** or just boost, since all it does is "boost" us from one frame to another traveling at constant velocity relative to the first, without any rotation. The general one involves some rotation as well. If the transformation is a *pure* rotation (no relative motion, just a change of orientation)m then $t' = t$. Thus pure rotation:

$$\Lambda = \Lambda_{\mathrm{R}} = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathbf{R} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

where $\mathbf{R}$ is the 3×3 matrix of the given rotation.

Four-Vector

In each inertial frame $\mathcal{S}$, a four-vector is specified by a set of four numbers $q = (q_1, q_2, q_3, q_4)$ such that the values in two frames $\mathcal{S}$ and $\mathcal{S}'$ are related by the equation $q' = \Lambda q$, where Λ is the Lorentz transformation connecting $\mathcal{S}$ and $\mathcal{S}'$.

Remark:

The great merit of the notion of four-vectors is that it often allows one to check with almost no effort whether a proposed physical law is relativistically invariant.

Any single quantity that is invariant under rotations is called a **rotational scalar** or a **three-scalar**; for example, the mass m of an object is a three-scalar, and so is the time t. In the same way, any single quantity that is invariant under Lorentz transformations is called a **Lorentz scalar** or **four-scalar**.

15.9 The Invariant Scalar Product

$$\begin{aligned} s' &= x_1'^2 + x_2'^2 + x_3'^2 - x_4'^2 \\ &= \gamma^2 \left(x_1 - \beta x_4\right)^2 + x_2^2 + x_3^2 - \gamma^2 \left(-\beta x_1 + x_4\right)^2 \\ &= \gamma^2 \left(1 - \beta^2\right) x_1^2 + x_2^2 + x_3^2 - \gamma^2 \left(1 - \beta^2\right) x_4^2 \\ &= s \end{aligned}$$

Note that we have $\gamma^2(1 - \beta^2) = 1$.

So we define an **invariant scalar product** in four-space. For any two four-vectors x, y, we define

$$x\dot{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$$

Therefore we can say that for any Lorentz transformation Λ,

$$x\dot{y} = (\Lambda x)\dot{(}\Lambda y)$$

15.10 The Light Cone

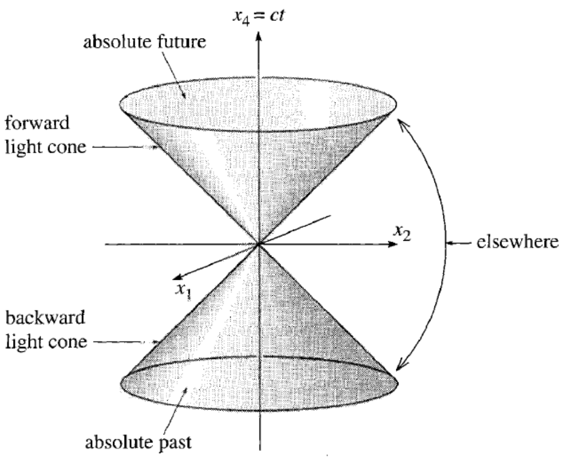


Figure 15.8 The light cone is defined by the condition $x \cdot x = r^2 - c^2t^2 = 0$ and divides space–time into five distinct parts: the forward and backward light cones, with $t > 0$ and $t < 0$ respectively; the interiors of the forward and backward light cones, called the absolute future and the absolute past; and the outside of the cone, labeled "elsewhere."

Figure 15.3: Figure 15.8 in textbook

To help visualize this, it is convenient to ignore one of the spatial dimensions (x_3 say), so we can plot the remaining two spatial dimensions horizontally and $x_4 = ct$ vertically up, as in Figure 15.3.

Interior of the Light Cone; Future and Past

Consider a space-time point P, with coordinates $x = (\mathbf{x}, ct)$, that lies inside the forward light cone. $t > 0$ and $r^2 < c^2t^2$ or

$$\begin{cases} x_4 > 0, \text{ and} \\ x_1^2 + x_2^2 + x_3^2 < x_4^2 \qquad (\text{or } x\dot{x} < 0) \end{cases}$$

Remark:

These two conditions have a remarkable consequence.

- any event occurs at P is later than any event at O, in the frame $\mathcal{S}$ in which our coordinates x are measured.
- both conditions hold in all frames if they hold in one frame. In other word, P lies inside the forward light cone is a Lorentz-invariant statement. For this reason, it is often called **absolute future**.

Similarly, we can have a light cone with its vertex at Q. Any point P on this cone must satisfy

$$(x_P - x_Q)^2 = 0$$

Exterior of the Light Cone; Space-Like Vectors

The situation is entirely different for a point P that lies outside the light cone. Condition:

$$(\mathbf{x}_P - \mathbf{x}_Q)^2 > c^2(t_P - t_Q)^2$$

or, equivalently

$$(x_P - x_Q)^2 > 0$$

This condition is symmetric between P and Q.

Proposition 15.1

Let P be any given space-time point outside the light cone of a second given point Q. Then

1. there exist frame $\mathcal{S}$ in which $t_P > t_Q$
   but
2. there also exist frames $\mathcal{S}'$ in which $t'_P = t'_Q$
   and
3. there also exist frames $\mathcal{S}''$ in which $t''_P < t''_Q$

Remark:

This startling proposition implies that the time ordering of any two given events, each outside the other's light cone, can be different in different frames: Where one observer says that event A occurred before event B, a second observer can find them the other way around (and a third can find them to be simultaneous). This has profound implications related to the notion of **causality**: If one event A (an explosion, for instance) is the cause of another event B (the collapse of a distant building), then A must obviously occur first in time, since causes always precede their effects. ... Therefore, nothing that happens at Q can be the cause of anything that happens at P, nor the other way around.

No causal influence can travel faster than the speed of light. Because the region outside the light cone of Q is completely immune to anything that happens at Q, this region is sometimes called the "elsewhere" of Q.

A four-vector whose fourth component is zero can be described as a pure-space vector, and one which can be brought into this form by a Lorentz transformation is called space-like.

With this terminology, we can say that the outside of the light cone is made up all space-like vectors.

Time-Like Vectors

Similarly, if a four-vector q lies inside the light cone ($q \cdot q < 0$), then there exists a frame $\mathcal{S}'$ in which it has the pure-time form $q' = (0, 0, 0, q'_4)$. Naturally, therefore, we describe vectors inside the light-cone as being **time-like**.

Equivalently (as we shall see), it is any particle for which, at any given time, there exists a rest frame; that is, a frame in which the particle is at rest, with $v = 0$.

Remark:

Final Exam Content Ends Here

15.11 The Quotient Rule and Doppler Effect

The Quotient Rule

Suppose that x is known to be a four-vector and that, in every inertial frame, $k = (k_1, k_2, k_3, k_4)$ is a set of four numbers, and suppose further that for every value of x the quantity $\phi = k \cdot x = k_1x_1 + k_2x_2 + k_3x_3 - k_4x_4$ is found to have the same value in all frames, then k is a four-vector.

Doppler Effect

Any sinusoidal plane wave has the form

$$\phi = A\cos(\mathbf{k} \cdot \mathbf{x} - \omega t - \sigma)$$

where $\mathbf{k}$ is called the wave vector, and $|\mathbf{k}| = 2\pi/\lambda$. Here we let $x = (\mathbf{x}, ct)$, then

$$\mathbf{k} \cdot x - \omega t = k \cdot x$$

and k denotes the wave four-vector

$$k = (\mathbf{k}, \omega/c)$$

$$\vdots$$

relativistic Doppler formular for light

$$\omega = \frac{\omega_0}{\gamma(1 - \beta\cos\theta)}$$

15.12 Mass, Four-Velocity, and Four-Momentum

Mass in Relativity

invariant mass and variable mass.

Invariant Mass

The mass, m, of an object, whatever its speed, is defined to be its rest mass.

Remark:

It is a Lorentz scalar.

The Proper Time of a Body

world line

The three-dimensional position $\mathbf{x}(t)$ of a body at time t defines a point $x = (\mathbf{x}(t), ct)$ in space—time, and, as time advances, this point traces a path, called the body's world line.

There is a frame (the body's rest frame) where the separation is pure time-like, with $dx_o = (0, 0, 0c\ dt_o)$. The subscript "o" indicates the rest frame.

$$dt_o = dt\sqrt{1 - v^2/c^2} = \frac{dt}{\gamma(v)}$$

The Four-Velocity

We may as well consider the four-vector

$$u = \frac{dx}{dt_o} = \left(\frac{d\mathbf{x}}{dt_o}, c\frac{dt}{dt_o}\right)$$

Since this four-velocity is the quotient of a four-vector and a four-scalar, it clearly is a four-vector. Then we replace dt_o by dt/γ, we find that

$$u = \gamma\left(\frac{d\mathbf{x}}{dt}, c\frac{dt}{dt}\right) = \gamma(\mathbf{v}, c)$$

Remark:

The most prominent feature of this result is that the three-velocity v is not the spatial part of the four-velocity u (which is why I called the latter u rather than

v).

Relativistic Momentum

Instead of using the three-velocity $\mathbf{v}$, suppose we used the four-velocity u to define the four-momentum of any object of mass m as

$$p = mu = (\gamma m\mathbf{v}, \gamma mc)$$

three-momentum $\mathbf{p}$, as the spatial part of it

$$\mathbf{p} = m\mathbf{u} = \gamma m\mathbf{v}$$

$$\sum p_{fin} = \sum p_{in} \tag{15.5}$$

Remark:

Since the four-momentum p is a four-vector, the same is true of both sides of (15.5). Therefore, if (15.5) is true in one frame $\mathcal{S}$, it is automatically true in all frames; that is, our proposed law of conservation of four momentum is compatible with the postulates of relativity.

Variable Mass

Some physicists like to rewrite the definition of the relativistic three momentum by introducing a variable mass

$$m_{var} = \gamma(v)m$$

with this defn the three-momentum becomes

$$\mathbf{p} = m_{var}\mathbf{v}$$

Remark:

Pros make the relativistic momentum look like its nonrelativistic counterpart

Cons First, it is not necessarily a good idea to make a new definition look like its older counterpart when there are, in fact, important differences. Second, the introduction of the variable mass fails to achieve a complete parallel with classical mechanics. Third, unlike the invariant mass, the variable is not a Lorentz scalar. For all of these reasons, I shall not use the variable mass here.

15.13 Energy, the Fourth Component of Momentum

Relativistic Energy

The energy E of a freely moving object with four-momentum $p = (\mathbf{p}, p_4)$ is

$$E = p_4 c = \gamma m c^2$$

With this defn,

$$p = (\mathbf{p}, E/c)$$

which explains why the four-momentum p is also called the momentum-energy four-vector.

With $v \ll c$,

$$\gamma = [1 - (v/c)^2]^{-1/2} = 1 + \frac{1}{2}(v/c)^2 + \dots$$

so the energy

$$E \approx mc^2 + \frac{1}{2}mv^2$$

Mass Energy

let us look again at the relativistic definition of the energy of an object, $E = \gamma mc^2$. Even if the object is at rest, with $\gamma = 1$, the object still has some energy, given by $E = mc^2$ (perhaps the most famous equation in all of physics). This energy is naturally called the **rest energy** of the object or, since it is associated with the mass m, the **mass energy**.

Three Useful Relations

$$\boldsymbol{\beta} \equiv \frac{v}{c} = \frac{\mathbf{p}c}{E}$$

$$p \cdot p = -(mc)^2$$

$$E^2 = (mc^2)^2 + (\mathbf{p}c)^2$$

15.14 Collisions

Skipped

Threshold Energies

A quantity of great concern to any experimenter hoping to observe this kind of reaction is the threshold energy.

15.15 Force in Relativity

Like most introductory texts, I shall avoid the complication of such "heat-like forces" by confining attention to forces that do not change the rest masses of the objects on which they act.

Of the several conceivable definitions of force in relativity, the single most useful is probably the **three-force** defined as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Remark:

- not the same as the nonrelativistic force
- with this defn, the force on a charge $q$in an electromagnetic filed is given by the Lorentz equation

$$\mathbf{F} = q(E + v \times B)$$

- we can prove the analog of the work-KE theorem

$$dT = \mathbf{F} \cdot d\mathbf{x}$$

  by differentiating $E^2 = (\mathbf{p}c)^2 + (mc^2)^2$.

Potential Energy

It can happen that, at least in one frame $\mathcal{S}$, the force $\mathbf{F}$ on an object is the gradient of a function $U(\mathbf{x})$.

The Four-Force

We can define the **four-force**

$$K = \frac{dp}{dt_o}$$

Remark:

There is no widely accepted notation for the four-force, buy K is one of the several notations used.

$$K = (\mathbf{K}, K_4) = \gamma(\mathbf{F}, \mathbf{v} \cdot \mathbf{F}/c)$$

The main disadvantage of the four-force is that it gives the time derivative of momentum with respect to the proper time, where the three-force gives the derivative with respect to the time of any one inertial frame.

15.16 Massless Particles; the Photon

Let's look at the two relations

$$E^2 = \left(mc^2\right)^2 + (\mathbf{p}c)^2 \quad \text{and} \quad \frac{\mathbf{v}}{c} = \frac{\mathbf{p}c}{E}$$

If $m = 0$, then we have

$$E = |\mathbf{p}|c \qquad \text{and} \qquad v = c$$

The photon is the particle that carries the energy and momentum of electromagnetic waves; and experiment shows that, for a photon, E and p do satisfy eqs above and that photons do always travel (no surprise!) at the speed of light.

With $m = 0$, the four-momentum of a photon satisfies

$$p^2 = 0$$

We have seen that the four-momentum of a material particle (that is, a particle with $m > 0$) is always forward time-like. By contrast, that of any massless particle lies on the forward light cone and is **forward light-like**.

In fact there is a second way to find the energy and momentum of the photon. One of the first discoveries (due to Max Planck and Einstein) in the unfolding of quantum mechanics was that the energy of a photon is related to the frequency of its associated electromagnetic wave by the famous relation

$$E = \hbar\omega$$

where $\hbar$ is Planck's constant ($\hbar = h/2\pi$) and omega is the angular freq of the wave. Similarly

$$\mathbf{p} = \hbar\mathbf{k} \tag{15.6}$$

where $\mathbf{k}$ is the wave vector of the wave. Since $p = (\mathbf{p}, E/c)$ and the wave four-vector is $k = (\mathbf{k}, \omega/c)$, the two relations imply that

$$p = \hbar k$$

The relation (15.6) is often rewritten in terms of the wavelength λ. Since $|\mathbf{k}| = 2\pi/\lambda$,

$$|\mathbf{p}| = \hbar|\mathbf{k}| = \frac{h}{\lambda}$$

which is often called the **de Broglie relation, 德布罗意方程**. I'll rewrite a photon's four-momentum as follows:

$$p = \hbar k = \hbar\left(\mathbf{k}, \frac{\omega}{c}\right) = \frac{\hbar\omega}{c}(\hat{\mathbf{k}}, 1)$$

The Compton Effect

康普顿效应

15.17 Tensors

Skipped.

15.18 Electrodynamics and Relativity

Skipped.

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16

Continuum Mechanics

Not studied in AMATH 271.

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