Fields and Galois Theory

PMATH 348

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Preface

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Sibelius Peng

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Introduction

Missed first 15 minutes...

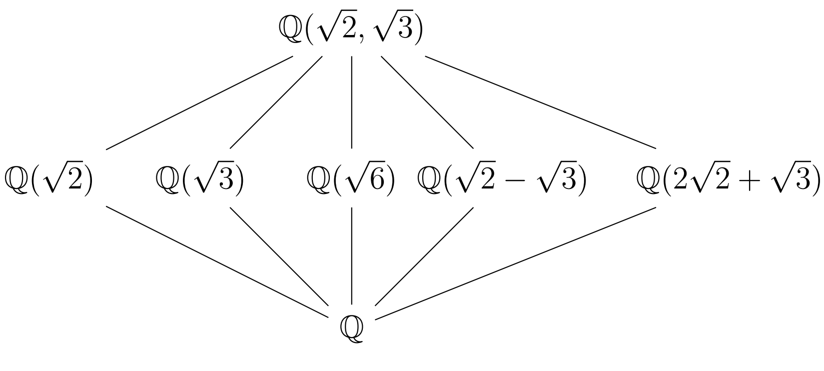
Quadratic eqns, Cubic eqns ...

Two main steps of Galois Theory

1. link a root of a quintic eqn, say α to $\mathbb{Q}(\alpha)$ the smallest field containing $\mathbb{Q}$ and α.

    - $\mathbb{Q}(\alpha)$ is a field so it has more structures to be played with than α.

    - However, our knowledge about $\mathbb{Q}(\alpha)$ is still too little to answer the question[1]. For example, we do not know how many intermediate filed E between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$, i.e. $\mathbb{Q} \subseteq E \subseteq \mathbb{Q}(\alpha)$. Note that



2. Link the field $\mathbb{Q}(\alpha)$ to a group. More precisely, we associate to field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ to the group

$$\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha)) = \{\phi = \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha) \text{ is an isomorphism and } \phi|_{\mathbb{Q}} = 1_{\mathbb{Q}}\}$$

    - It can be shown that if α is 'good', say algebraic. $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha))$ is finite.

[1] is raised in first 15 mins...

- If α is 'very good' say constructible[2], the order of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha))$ is in certain forms.

- Moreover, there is 1-1 correspondence between the intermediate fields $\mathbb{Q}(\alpha)/\mathbb{Q}$ and the subgroups of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha))$.

Galois Theory the interplay between fields and groups.

[2]formal defn later