Graph Theory

CO 442

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LaTeXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of CO 442 during Fall 2020 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

Here is the notation used in this course.

- $\chi(G)$: chromatic number, k vertex coloring
- $\Delta(G)$: max degree of vertices
- $\delta(G)$: min degree of vertices
- $\omega(G)$: max size of a clique
- $\chi'(G)$: chromatic index, edge chromatic number, k edge coloring
- $\alpha(G)$: independence number, size of the maximum independent set
- $L(G)$: line graph of G
- $\chi_\ell(G)$: list chromatic number
- $\chi'_\ell(G)$: list chromatic index
- $\mu(G)$: multiplicity of a multigraph G

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Sibelius Peng

Contents

First let's look at a proof example.

Theorem

Every two longest paths in a connected graph G intersect.

Proof:

Suppose note. That is, there exist two longest paths P_1 and P_2 of G such that $V(P_1) \cap V(P_2) = \varnothing$. For each $i \in \{1,2\}$, let $v_{i,1}$ and $v_{i,2}$ be the ends of P_i. Since G is connected, there exists a shortest path P from $V(P_1)$ to $V(P_2)$. Since P is shortest, we have that $|V(P_i) \cap V(P)| = 1$ for each $i \in \{1,2\}$.

For each $i \in \{1,2\}$, let u_i be the end of P in $V(P_i)$. For each $i, j \in \{1,2\}$, let $Q_{i,j}$ be the subpath of P_i from u_i to $v_{i,j}$. We assume without loss of generality that for each $i \in \{1,2\}$, we have that $|E(Q_{i,1})| \geq |E(Q_{i,2})|$ and hence

$$|E(Q_{i,1})| \geq |E(P_i)|/2.$$

Let $P' = v_{1,1}Q_{1,1}u_1Pu_2Q_{2,1}v_{2,1}$. Note that P' is a path in G and

$$|E(P')| = |E(Q_{1,1})| + |E(P)| + |E(Q_{2,1})| \geq |E(P)| + |E(P_1)| > |E(P_1)|.$$

Hence P' is a longer path than P_1, contradicting that P_1 is a longest path. $\square$

Things to remember:

1. Correctness
2. Clarity/Precision
3. Ease of Reading

1

Colorings

1.1 Coloring and Brooks' Theorem

coloring

A **coloring** of a graph G is an assignment of colors to vertices of G such that no two adjacent vertices receive the same color.

k-coloring

Let G be a graph. We say $\phi : V(G) \to [k]$ is a k**-coloring** of G if $\phi(u) \neq \phi(v)$ for every $uv \in E(G)$.

Since every graph G has a $|V(G)|$-coloring, we are interested in the minimum numbers of colors needed to color G.

chromatic number

The **chromatic number** of a graph G, denoted $\chi(G)$, is the minimum number k such that G has a k-coloring.

Then why coloring?

- Coloring is a foundational problem in graph labeling, wherein we study functions on $V(G)$ according to constraints imposed on the graph (e.g. non-adjacent vertices are labeled differently)
- Coloring is a foundational problem in graph decomposition, wherein we seek to decompose $V(G)$ into certain kinds of subgraphs (e.g. independent sets)
- Applications to maps, scheduling, job processing, frequency assignment (e.g. cell networks)
- Applications to algorithms (e.g. distributed computing)

However, coloring is hard.

A graph being an **independent set** is by definition equivalent to being **1-colorable**.

A graph being **bipartite** is by definition equivalent to being **2-colorable**. (Indeed coloring is a generalization of partite)

Proposition 1.1

G is 2-colorable if and only if G does not contain an odd cycle.

Moreover, there exists a poly-time algorithm to decide if G is 2-colorable.

Theorem: Karp (1972)

For each $k \geq 3$, deciding if a graph G has a k-coloring is NP-complete.

Indeed, 3-coloring is NP-complete even for planar graphs. Any constant factor approximation is also NP-complete.

Then what about the bounds on chromatic number?

As mentioned $\chi(G) \leq |V(G)|$.

Greedy Upper bound: $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of vertices in G. Why? By a greedy algorithm:

- Order the vertices of G arbitrarily, $v_1, \ldots, v_{|V(G)|}$.
- Color the vertices in order avoiding the colors of previously colored neighbors.
- Since each vertex has at most $\Delta(G)$ neighbors, there is always at least one color for the current vertex.

Lower bound: $\chi(G) \geq \omega(G)$, where $\omega(G)$ denotes the clique number of H, that is the maximum size of a clique in G.

Can we do better than the greedy upper bound?
No! The bound is tight for complete graphs: $\omega(K_n) = \chi(K_n) = (n-1) + 1 = \Delta(K_n) + 1$.

Can we do better if the graph is not complete?
No! The graph could have a component that is complete.

Can we do better if the graph is connected and not complete?
No! The bound is tight for odd cycles: $\chi(C_{2k+1}) = 3 = 2 + 1 = \Delta(C_{2k+1}) + 1$.

Can we do better if the graph is connected and neither complete nor an odd cycle? **Yes!**

Theorem 1.2: Brooks 1941

If G is connected, then $\chi(G) \leq \Delta(G)$ if and only if G is neither complete nor an odd cycle.

An Informal Proof of Brooks' Theorem

How to prove Brook's Theorem?

Actually there are 8 to 10 distinct ways to prove Brooks' Theorem. See the nice survey *Brooks' Theorem and Beyond* by Cranston and Rabern from 2014 for more details. Here are some of those methods: Greedy Coloring, Kempe Chains, List Coloring, Alon-Tarsi Theorem, Kernel Perfection, Potential Method.

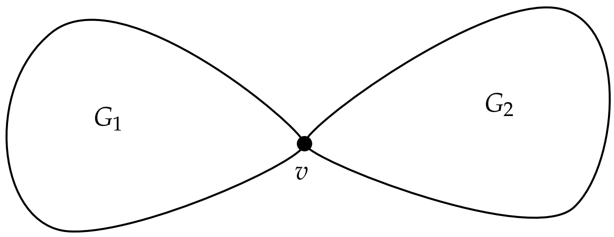
Today we give an informal proof sketch via the Greedy Coloring Method - arguably the most direct, brute-force of the approaches. (See Diestel for the Kempe Chain proof).

The idea is to try a method (greedy coloring) we know works for a similar problem ($\Delta + 1$-coloring), and ask under what conditions can we use this to get the desired outcome (a Δ-coloring).

In the other cases we cannot apply greedy, we instead do **reductions**: that is, we show how to inductively color or to show that the graph is one of the exceptional outcomes (clique or odd cycle).

Alternatively, we could have built up a suite/library of reductions that work, and then tried to find a method to deal a finishing blow (i.e. to handle the cases we could not reduce).

First Reduction G has a cutvertex v. Then v separates G into two smaller graphs G_1 and G_2.



$\phi_1(v) = \phi_2(v)$

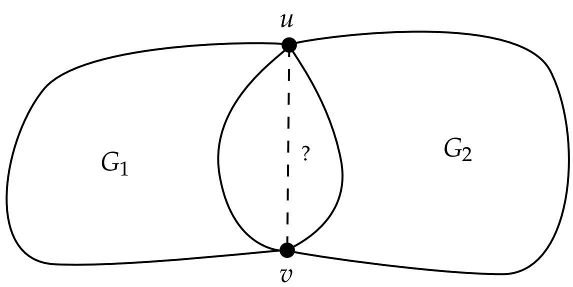
By minimality of G, G_i has a Δ-coloring ϕ_i, $i \in 1, 2$.

This only works if neither graph is $K_{\Delta+1}$ or odd cycle when $\Delta = 2$.

Now permute the colors in ϕ_2 so that $\phi_1(v) = \phi_2(v)$. Then $\phi_1 \cup \phi_2$ yields a Δ-coloring of G, a contradiction.

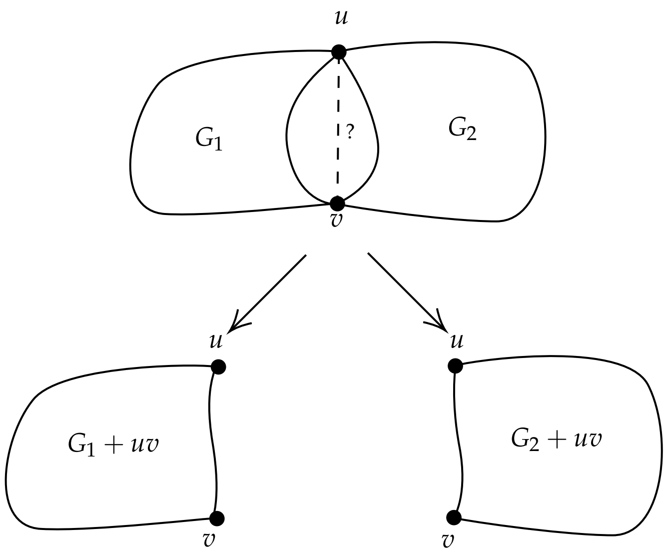
Second Reduction G has a cutset $\{u, v\}$.

Try the same trick. Say $\{u, v\}$ separates G into two smaller graphs G_1 and G_2. By induction or minimum counterexample, each of G_1, G_2 has a Δ-coloring ϕ_i, $i \in 1, 2$.



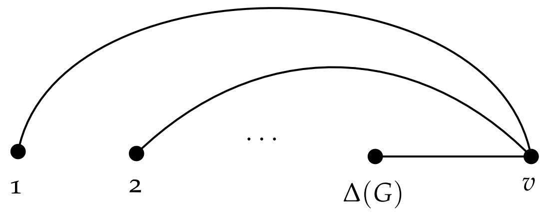
If $uv \in E(G)$, then we can permute the colorings so that $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$.

This fails if $uv \notin E(G)$. Because we may have u, v colored the same in one coloring and different in the order and no permuting will fix this! So we can add the edge uv to both G_1 and G_2!

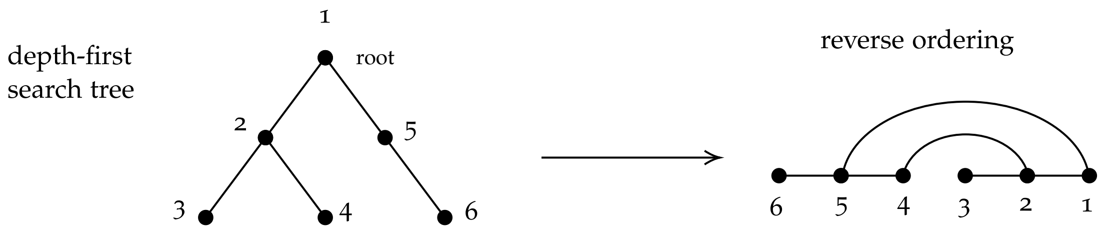


Have to show $\Delta(G_1 + uv), \Delta(G_2 + uv) \leq \Delta(G)$. We also have to ensure that neither G_1 nor G_2 is complete (or odd cycle in $\Delta(G) = 2$ case).

Then we assume G is 3-connected. We now turn to the finishing blow (greedy). The greedy *fails* when a vertex has $\Delta(G)$ earlier neighbors in the ordering, each with a different color from $\{1, \ldots, \Delta(G)\}$.

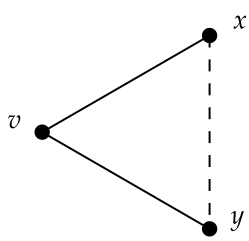


Can we find an ordering where most of the vertices have at most $\Delta(G) - 1$ earlier neighbors? Yes for all but the last vertex in the ordering! We can fix a root, then take a depth-first search tree ordering from the root. Reverse it!

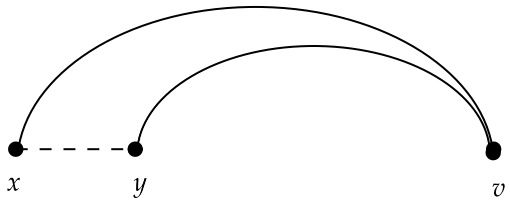


Now all vertices but the last will be fine in greedy.

If $\deg(v) \leq \Delta(G) - 1$, then we can ensure greedy does not fail at the last vertex v. Otherwise, we ensure that two of its neighbors x and y are colored the same (and hence there is a color left for v when it is v's turn). These two are two non-adjacent neighbors, which guaranteed to exist as G is not $K_{\Delta+1}$.



We can put x, y first in the ordering to guarantee x and y are colored the same. Then we can color them as we desire (since non-adjacent), say both with color 1.



$\phi(x) = 1 = \phi(y)$

Use the reverse of a depth-first search tree ordering of $G - \{x, y\}$ with root v, then we finish the ordering so every vertex in $V(G) \setminus \{x, y, v\}$ has at most $\Delta(G) - 1$ earlier neighbors. Since $G - \{x, y\}$ is connected as G is 3-connected, then this ordering exist.

A Formal Proof of Brooks' Theorem

Let us codify our ordering fact as a proposition.

Proposition 1.3: Ordering Proposition

If G is a connected graph on n vertices and $v \in V(G)$, then there exists an ordering $v_1, \ldots, v_n = v$ of $V(G)$ such that $|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \geq 1$ for all $i \in [n-1]$.

Proof:
Reverse a depth-first search tree ordering from root v. Or more formally:

We proceed by induction on $|V(G)|$. If $|V(G)| = 1$, then the ordering v is as desired. So we assume that $|V(G)| \geq 2$. Let $G_1, \ldots, G_k$ be the components of $G - v$. As G is connected, there exists neighbors $u_1, \ldots, u_k$ of v such that $u_i \in V(G_i)$ for each $i \in [k]$. For each $i \in [k]$, there exists by induction applied to G_i and u_i, an ordering σ_i of $V(G_i)$ as prescribed by the proposition. Let σ be the ordering of $V(G)$ obtained by concatenating the σ_i and finally v. Then σ is as desired. $\square$

Now we are ready to prove Brooks' Theorem:

Suppose not. Let G a counterexample with $|V(G)|$ minimized. If $\Delta(G) \leq 2$, the result is standard. So we assume that $\Delta(G) \geq 3$.

Claim 1 There does not exist a cutvertex of G.

Proof:
Suppose not. That is, there exists a cutvertex v of G and two connected subgraphs G_1, G_2 of G such that $G_1 \cap G_2 = \{v\}, G_1 \cup G_2 = G$ and $|V(G_i)| < |V(G)|$ for each $i \in [2]$.

As G_1 and G_2 are subgraphs of G, we have that $\Delta(G_i) \leq \Delta(G)$ for each $i \in [2]$. Moreover, as G is connected, we have for each $i \in [2]$ that $\deg_{G_i}(v) \geq 1$ and hence $\deg_{G_i}(v) \leq \Delta(G) - 1$. Hence $G_i \neq K_{\Delta(G)+1}$ for each $i \in [2]$. Thus by the minimality of G, there exist $\Delta(G)$-colorings ϕ_i of G_i for each $i \in [2]$.

By permuting the colors of ϕ_2 as necessary, we assume without loss of generality that $\phi_1(v) = \phi_2(v)$. But then $\phi_1 \cup \phi_2$ is a $\Delta(G)$-coloring of G, a contradiction. $\square$

Claim 2 There does not exist a 2-cut of G, or, there exists a vertex $v \in V(G)$ with $\deg_G(v) \leq \Delta(G) - 1$.

Proof:
Suppose not. Now let us suppose there exists a 2-cut $\{v_1, v_2\}$ of G and two connected subgraphs G_1, G_2 of G such that $G_1 \cap G_2 = \{v_1, v_2\}$, $G_1 \cup G_2 = G$ and $|V(G_i)| < |V(G)|$ for each $i \in [2]$.

Choose v_1, v_2, G_1, G_2 such that neither $G_1 + v_1v_2$ nor $G_2 + v_1v_2$ is equal to $K_{\Delta(G)+1}$ if possible.

As G is connected and G does not have a cutvertex by Claim 1, we have for all $i, j \in [2]$ that $\deg_{G_i}(v_j) \geq 1$ and hence $\deg_{G_i}(v_j) \leq \Delta(G) - 1$. Thus $\Delta(G_i + v_1v_2) \leq \Delta(G)$ for all $i \in [2]$.

Next suppose that there exists $i \in [2]$ such that $G_i + v_1v_2 = K_{\Delta(G)+1}$. Without loss of generality, we assume that $i = 1$. Let v_1' be the neighbor of v_1 in $G_2 - v_2$. Let $G_1 = G_1 + v_1v_1'$ and $G_2' = G_2 \setminus \{v_1\}$. Now wither $\deg_G(v_1') \leq \Delta(G) - 1$, a contradiction, or we find that $G_i' + v_1'v_2 \neq K_{\Delta(G)+1}$ for each $i \in [2]$. But then v_1', v_2, G_1', G_2' contradict the choice of v_1, v_2, G_1, G_2.

So we assume that $G_1 + v_2v_2, G_2 + v_1v_2 \neq K_{\Delta(G)+1}$. Thus by the minimality of G, there exist $\Delta(G)$-colorings ϕ_i of G_i for each $i \in [2]$. By permuting the colors of ϕ_2 as necessary, we assume without loss of generality that $\phi_1(v_j) = \phi_2(v_j)$ for each $j \in [2]$. But then $\phi_2 \cup \phi_2$ is a $\Delta(G)$-coloring of G, a contradiction. $\square$

Let $v \in V(G)$ with $\deg_G(v)$ minimized.

First suppose that $\deg_G(v) \leq \Delta(G) - 1$. By the Ordering Proposition, there exists an ordering $v_1, \ldots, v$ of $V(G)$ such that $|N(v_i) \cap \{v_{i+1}, \ldots, v\}| \geq 1$ for all $i \in [|V(G)| - 1]$. Now greedily color $V(G)$ in that order. This yields a $\Delta(G)$-coloring of G, a contradiction.

So we assume that $\deg_G(v) = \Delta(G)$. Since $G \neq K_{\Delta+1}$, there exist distinct $x, y \in N(v)$ such that $xy \notin E(G)$. By Claims 1 and 2, it follows that G is 3-connected and hence $G - \{x, y\}$ is connected. Hence by the Ordering Proposition, there exists an ordering $v_1, \ldots, v$ of $V(G) - \{x, y\}$ such that $|N(v_i) \cap \{v_{i+1}, \ldots, v\}| \geq 1$ for all $i \in [|V(G)| - 3]$. Now color x, y with color 1. Then greedily color $V(G) - \{x, y\}$ in that order. This yields a $\Delta(G)$-coloring of G, a contradiction.

Beyond Brooks' Theorem

Can we go further? Can we save more colors? Under what conditions?

Question (ω, Δ, χ paradigm)

What is the maximum chromatic number of graphs with $\omega(G) \leq \omega$ and $\Delta(G) \leq \Delta$?

Brooks' Reformulated

If G is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.

Borodin-Kostochka Conjecture (1977)

If G is a graph with $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G) - 1$, then $\chi(G) \leq (G) - 1$.

Why $\Delta \geq 9$?

Let $G = C_5 \boxtimes K_3$. (the blowup of every vertex in C_5 to a triangle K_3) Then $\Delta(G) = 8$, $\omega(G) = 6$, and yet $\chi(G) = 8$.

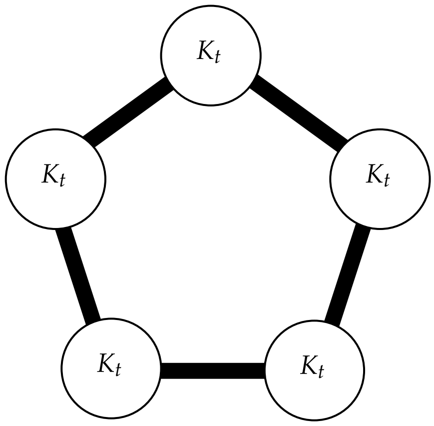
Theorem (Reed 1999)

True for $\Delta(G) \geq 10^{14}$.

Reed's conjecture

Reed's Conjecture (1998)

$$\chi(G) \le \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil.$$



5-cycle blowup

$\Delta = 3t-1$
$\omega = 2t$

$$\left\lceil \frac{1}{2}(\Delta+1+\omega) \right\rceil = \left\lceil \frac{5t}{2} \right\rceil$$

$\alpha = 2$

Theorem (Reed 1998)

The conjecture holds when $\Delta(G)$ is sufficiently large and

$$\omega(G) \ge (1-7\cdot 10^{-7})\Delta(G).$$

Corollary (Reed)

There exists $\varepsilon > 0$ such that for every graph G,

$$\chi(G) \le (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G).$$

Reed's value of ε was 10^{-8}.

Can we improve the ε for large enough Δ? Can we get closer to $\varepsilon = 1/2$?

For Large enough Δ, the following ε suffices:

- $\frac{1}{320e^6}$ (King and Reed 2012)
- $\frac{1}{26}$ (Bonamy, Perrett, Postle 2016+)
- $\frac{1}{13}$ (Delcourt and Postle 2017+)
- $\frac{1}{8.4}$ (Hurley, de Joannis de Verclos, Kang 2020+)

Large Girth

The **girth** of a graph G is the length of a shortest cycle in G.

Theorem (Erdős 1959)

$\forall g, k \ge 1$, there exists graphs of girth at least g and chromatic number at least k.

Theorem (Frieze and Luczak 1992)

Random d-regular graphs have chromatic number $(1-o(1))\frac{d}{2\ln d}$ with high probability.

Corollary

$\forall g, d \geq 1$, there exists a d-regular graph G of girth at least g with

$$\chi(G) \geq (1-o(1))\frac{d}{2\ln d}.$$

Girth-Five and Triangle-Free

Theorem (Kim 1995)

If G is a graph of girth five, then

$$\chi(G) \leq (1+o(1))\frac{\Delta(G)}{\ln \Delta(G)}.$$

Theorem (Johansson 1996)

If G is a triangle-free graph, then

$$\chi(G) \leq O\left(\frac{\Delta(G)}{\ln \Delta(G)}\right).$$

Theorem (Molloy 2017)

If G is a triangle-free graph, then

$$\chi(G) \leq (1+o(1))\frac{\Delta(G)}{\ln \Delta(G)}.$$

Small Clique Number

Theorem (Johansson 1999)

For every fixed r: if G is a graph with $\omega(G) \leq r$, then

$$\chi(G) \leq O\left(\frac{\Delta(G)}{\ln \Delta(G)} \cdot \ln\ln \Delta(G)\right).$$

Theorem (Molloy 2017)

$$\chi(G) \leq 200 \cdot \omega(G) \cdot \frac{\Delta(G)}{\ln \Delta(G)} \cdot \ln\ln \Delta(G).$$

Good for $\omega(G) \leq \frac{\ln \Delta(G)}{\ln\ln \Delta(G)}$. What if $\omega(G)$ is larger?

Question

For $k \geq 2$, what value of $\omega(G)$ guarantees $\chi(G) \leq \frac{\Delta(G)}{k}$?

Theorem (Bonamy, Kelly, Nelson, Postle 2018+)

$$\chi(G) \leq O\left(\Delta(G) \cdot \sqrt{\frac{\ln \omega(G)}{\ln \Delta(G)}}\right).$$

Corollary

$\forall k \geq 2$, if $\omega(G) \leq \Delta(G)^{\frac{1}{(192k)^2}}$, then
$$\chi(G) \leq \frac{\Delta(G)}{k}.$$

Ramsey theory constructions show that we cannot extend this beyond $\Delta(G)^{\frac{2}{k-1}}$.

1.2 Edge Coloring

edge coloring

An **edge-coloring** of a graph G is an assignment of colors to edges of G such that no two incident edges receive the same color.

k edge coloring

Let G be a graph. We say $\phi : E(G) \to [k]$ is a k**-edge-coloring** of G if $\phi(e) \neq \phi(f)$ for every $e, f \in E(G)$ with $e \sim f$.

Here $e \sim f$ means e, f share a common endpoint ("are adjacent") in G.

chromatic index

The **chromatic index** of a graph G (also known as **edge chromatic number**), denoted $\chi'(G)$, is the minimum number k such that G has a k-edge-coloring.

line graph

The **line graph** of a graph G, denoted by $L(G)$, is the graph where $V(L(G)) := E(G)$ and $E(L(G)) := \{ef : e, f \in E(G), e \sim f\}$.

Edge colorings of G are equivalent to vertex colorings of $L(G)$. Hence $\chi'(G) = \chi(L(G))$.

What are some natural upper and lower bounds on $\chi'(G)$?

Proposition 1.4

$$\Delta(L(G)) \leq 2\Delta(G) - 2$$

Hence by greedy,

$$\chi'(G) \leq 2\Delta(G) - 1$$

Proposition 1.5

$$\omega(L(G)) \geq \Delta(G)$$

Hence

$$\chi'(G) \geq \Delta(G)$$

Moreover, $\omega(L(G)) = \Delta(G)$ if $\Delta(G) \geq 3$.

Note that for even cycles, namely K_n, n even, $\chi'(G) = \Delta(G)$. For odd cycles, we have $\chi'(G) > \Delta(G)$.

Theorem 1.6: König (1916)

If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.

Proof (first):

It suffices to prove the theorem when G is $\Delta(G)$-regular since every bipartite graph G is a subgraph of some $\Delta(G)$-regular graph H.

Prove by induction on $\Delta(G)$. If $\Delta(G) = 0$, then the statement holds trivially. So assume $\Delta(G) \geq 1$.

Let $S \subseteq A$. By double counting $E(G(S, N(S)))$, it follows that

$$\Delta(G)|S| = |E(G(S, N(S)))| \leq \Delta(G)|N(S)|,$$

and thus $|S| \leq |N(S)|$. Hence by Hall's theorem, there exists a perfect matching M of G.

By induction, $G - M$ has a $(\Delta(G) - 1)$-coloring ϕ. Let $\phi(e) = \Delta(G)$ for each $e \in M$. Then ϕ is a $\Delta(G)$-coloring of G as desired. $\square$

Kempe chain

Let ϕ be a partial k-edge-coloring of a graph G. If $a, b \in [k]$ and $v \in V(G)$, then (a, b)-chain at v in ϕ, denoted $P_v(a, b, \phi)$ is the maximal path/cycle of a and b colored edges containing v.

switching

The coloring ϕ' obtained from switching (aka **recoloring**) $P_v(a, b, \phi)$ is defined as:

- $\phi'(e) = \{a, b\} \setminus \phi(e)$ if $e \in P_v(a, b, \phi)$, and
- $\phi'(e) = \phi(e)$ otherwise.

missing colors

Let ϕ be a partial k-edge coloring of a graph G.

- A coloring $a \in [k]$ is missing at v in ϕ if $a \notin \{\phi(e) : e \sim v\}$.
- We let $\phi(v)$ denote the set of missing colors at v.

Proof (second proof of König):

We proceed by induction on $|E(G)|$. If $E(G) = \varnothing$, there is nothing to show. So we assume that $E(G) \neq \varnothing$.

let $e = uv \in E(G)$. By induction, there exists a $\Delta(G)$-edge-coloring of $G - e$. Let ϕ be a $\Delta(G)$-edge-coloring of $G - e$ such that $|\phi(u) \cap \phi(v)$ is maximized.

Note that $\phi(u), \phi(v) \neq \varnothing$ since $\deg_{G-e}(u), \deg_{G-e}(v) \leq \Delta(G) - 1$.

If $\phi(u) \cap \phi(v) \neq \varnothing$, then let $\phi(e) \in \phi(u) \cap \phi(v)$ and hence ϕ is a $\Delta(G)$-edge-coloring of G as desired.

Now assume that $\phi(u) \cap \phi(v) = \varnothing$. Let $a \in \phi(u), b \in \phi(v)$. Note that $P := P_u(a, b, \phi)$ is a path. If $v \in V(P)$, then it follows that P has even length and hence $P + e$ is an odd cycle in G, contradicting that G is bipartite.

So we assume that $v \notin V(P)$. But then switching P yields a coloring ϕ' such that $b \in \phi'(u) \cap \phi'(v)$, contradicting the choice of ϕ. $\square$

1.2.1 Vizing's Theorem

Theorem 1.7: Vizing (1964)

If G is a graph, then $\chi'(G) \leq \Delta(G) + 1$.

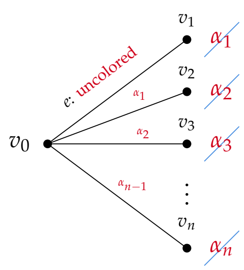
A graph G is called **class 1** if $\chi'(G) = \Delta(G)$, or **class 2** if $\chi'(G) = \Delta(G) + 1$. Note that deciding if a graph is class 1 is NP-complete!

Vizing fan

Suppose that G is a graph, $e = v_0 v_1 \in E(G)$, and ϕ is a partial k-edge-coloring of $G - e$ for some integer k. We say $T = (v_0 e_1 v_1 e_2 \ldots e_n v_n)$ is a **Vizing fan** with respect to the edge e, vertex v_0 and the coloring ϕ if

- $v_0, v_1, v_2, \ldots, v_n$ are all disjoint, and
- $\forall j, 1 \leq j \leq n, e_j = v_j v_0$, and
- $\forall j, 2 \leq j \leq n, \phi\left(e_j\right) \in \bigcup_{i<j} \phi\left(v_i\right)$.

Here is a depiction of Vizing fan.



Idea is no coloring implies disjoint missing colors in Vizing fan.

Lemma 1.8: Disjoint missing colors

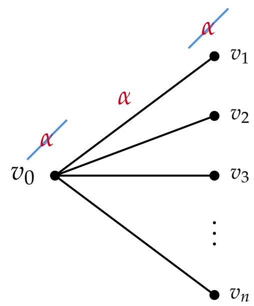
Let G be a graph and $e = v_0v_1 \in E(G)$ such that for some integer $k \geq \Delta(G) + 1$, $G - e$ has a k-edge-coloring, but G does not.

If ϕ is a k-edge-coloring of $G - e$, and $T = (v_0e_1v_1e_2v_2 \ldots e_nv_n)$ is a Vizing fan with respect e, v_0 and ϕ, then $\phi(v_i) \cap \phi(v_j) = \varnothing$ for all distinct $v_i, v_j \in V(T)$.

Proof:
Suppose not. That is $\exists i < j \in \{0, \ldots, n\}$ such that $\phi(v_i) \cap \phi(v_j) \neq \varnothing$.

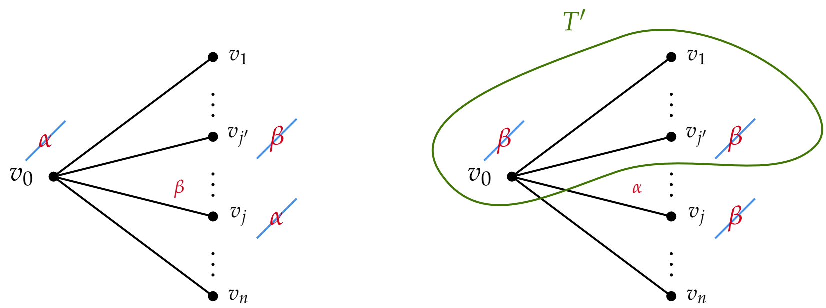
Let ϕ, T, i, j be chosen such that j is minimized, and subject to that condition, i is minimized. Let $\alpha \in \phi(v_i) \cap \phi(v_j)$. Three cases: $i = 0$ and $j = 1$; $i = 0$ and $j > 1$; $i > 0$.

Case 1: $i = 0, j = 1$.



Let $\phi(e) = \alpha$ and hence ϕ is a k-edge-coloring of G, a contradiction.

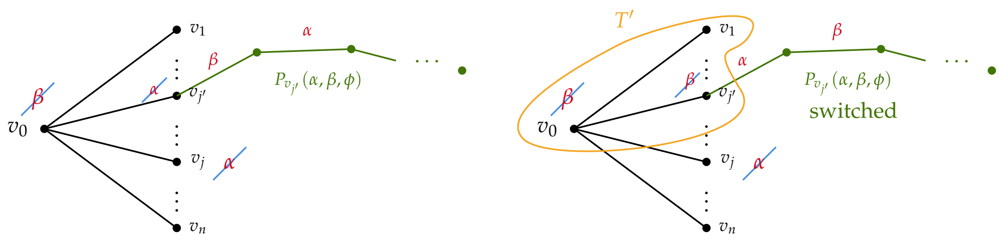
Case 2: Let $\beta = \phi(v_0v_j)$. Since T is a Vizing fan, $\exists j' < j$ such that $\beta \in \phi(v_{j'})$.



Let ϕ' be obtained from ϕ switching $P_{v_0}(\alpha, \beta, \phi) = v_0v_j$. Then $\beta \in \phi'(v_0) \cap \phi'(v_{j'})$. Now $T' := T[\{v_0, \ldots, v'_j\}]$ and ϕ' contradict minimality of T and ϕ.

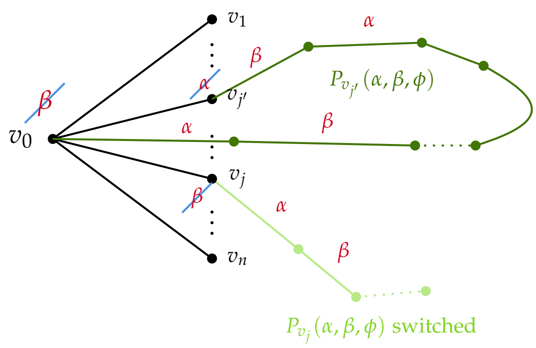
Case 3: $i > 0$. Let $\beta \in \phi(v_0)$. By minimality of T, $\beta \neq \alpha$. Let $j' := i$.

(a) $v_0 \notin V(P_{v_i}(\alpha, \beta, \phi))$.



Let ϕ' be obtained from ϕ by switching $P_{v_i}(\alpha, \beta, \phi)$. Then $\beta \in \phi'(v_0) \cap \phi'(v_i)$. Now $T' := T[\{v_0, \ldots, v_i\}]$ and ϕ' contradict the minimality of T and ϕ.

(b) $v_0 \in V(P_{v_i}(\alpha, \beta, \phi))$.



Let ϕ' be obtained from ϕ by switching $P_{v_j}(\alpha, \beta, \phi)$. Then $\beta \in \phi'(v_0) \cap \phi'(v_j)$. Now $T' := T[\{v_0, \ldots, v_j\}]$ and ϕ' contradict he minimality of T and ϕ.

□

Now we can take maximum Vizing fan and apply lemma for contradiction.

Proof of Vizing's Theorem:

Let G be a counterexample with $|V(G)|$ minimized. Hence $E(G) \neq \varnothing$. Let $e = v_0v_1 \in E(G)$. By the minimality of G, $G - e$ has $(\Delta(G) + 1)$-edge-coloring ϕ of $G - e$.

Let $T = (v_0e_1v_1e_2v_2 \ldots e_nv_n)$ be a Vizing fan with respect to e, v_0 and ϕ such that n is maximized. By lemma, $\phi(v_i) \cap \phi(v_j) = \varnothing$ for all distinct $i, j \in \{0, \ldots, n\}$.

Let $X := \bigcup_{i \in [n]} \phi(v_i)$. By Lemma, $|X| \geq n$. So $\exists \alpha \in X$ such that $\nexists f = v_0v_j$, $j \in [n]$ with $\phi(f) = \alpha$. Since $\alpha \notin \phi(v_0)$, $\exists e_{n+1} = v_0v_{n+1}$ with $\phi(e_{n+1}) = \alpha$. But then $T + e_{n+1}$ is a larger Vizing fan with respect to e, v_0, ϕ, contradicting the maximality of T. □

1.2.2 List Edge Coloring

List Coloring:

- list-assignment L: an assignment of lists $L(v)$ for $v \in V(G)$.
- k-list-assignment L: $|L(v)| \geq k$ for all $v \in V(G)$.
- L-coloring: a coloring ϕ where $\phi(v) \in L(v)$ for all $v \in V(G)$.
- A graph G is k-list-colorable if G has an L-coloring for every k-list-assignment L.

list chromatic number

The **list chromatic number**, denoted $\chi_\ell(G)$, is the minimum k such that G has an L-coloring for every k-list-assignment L.

Proposition 1.9

For all integer $k \geq 0$, there exists a bipartite graph G with $\chi_\ell(G) = k$.

Theorem (Alon 2000)

If G has average degree d, then $\chi_\ell(G) \geq \Omega(\log d)$.

For edge-coloring, we have something similar.

list chromatic index

The **list chromatic index**, denoted $\chi'_\ell(G)$, is $\chi_\ell(L(G))$.

List Coloring Conjecture (Various authors, 1970s/80s)

If G is a graph, then $\chi'_\ell(G) = \chi'(G)$.

Theorem 1.10: Galvin, 1995

If G is a bipartite graph, the $\chi'_\ell(G) = \chi'(G)$.

Theorem (Kahn, 1996)

If G is a graph, then $\chi'_\ell(G) = (1 + o(1))\chi'(G)$.

latin square

A **latin square** is an n by n array such that each of the numbers 1 to n appears exactly once in each row and exactly once in each column.

Equivalently, a latin square is an n-edge-coloring of $K_{n,n}$. Such always exist by König's theorem which shows that $\chi'(K_{n,n}) = \Delta(K_{n,n}) = n$.

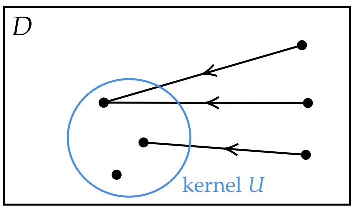
Dinitz Conjecture (1978)

Given an n by n array and an assignment of n symbols to each square, there exists a choice of symbol for each square such that each symbol appears at most once in each row and each column.

Or equivalently, by Galvin's Theorem, $\chi'_\ell(K_{n,n}) = n$.

kernel

A **kernel** of a digraph D is an independent set U such that every vertex in $D \setminus U$ has an out-neighbor in U.



kernel perfect

We say an orientation D of a graph G is **kernel perfect** if every induced subgraph D' of D has a kernel.

Lemma 1.11: Kernel Perfect Lemma

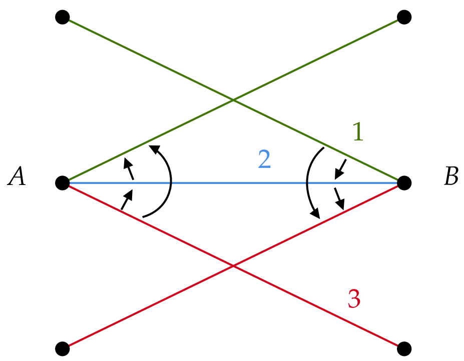
Let G be a graph and L a list assignment. If G has a kernel perfect orientation D such that $d^+(v) < |L(v)|$ for every v in D, then G has an L-coloring.

Proof:
By induction on $|V(G)|$. If $|V(G)| = 0$, we are done.

Let $v \in V(G)$, $\alpha \in L(v)$ and $D' := G[\{u \in V(G) : \alpha \in L(u)\}]$. Since D is kernel perfect, D' has a kernel U. Let $\phi(u) = \alpha$ for all $u \in U$, and $L'(x) := L(x) \setminus \{\alpha\}$ for all $x \in G \setminus U$. Now $d^+_{G\setminus U}(v) < |L'(v)|$ for all $v \in D \setminus U$. By induction, $G \setminus U$ has an L'-coloring and hence G has an L-coloring. □

Proof of Galvin's Theorem:
Let $G = (A, B)$. Let $k = \chi'(G) = \Delta(G)$. Let ϕ be a k-edge-coloring of G.

Let D be an orientation of $L(G)$ where $e \sim e' \in E(G)$ with $\phi(e) < \phi'(e)$, we orient ee': $e' \to e$ if $e \cap e' \in A$; $e \to e'$ if $e \cap e' \in B$.



Let $e = uv$, $u \in A, v \in B$. Let $\phi(e) = i$. Then

$$\begin{aligned} d^+(e) &\le \left|\{e' : e' \cap e = u, \phi(e') < \phi(e)\}\right| + \left|\{e' : e' \cap e = v, \phi(e') > \phi(e)\}\right| \\ &\le (i-1) + (\Delta(G) - i) = \Delta(G) - 1 = k - 1 \end{aligned}$$

Let D' be an induced subgraph of D. Then D' has a kernel U, namely a stable matching as guaranteed by the Stable Marriage Theorem (where v prefers u to u' if $u'v$ is directed towards uv). Hence D is kernel perfect. By Kernel Perfect Lemma, $L(G)$ has an L-coloring. □

Proof Ideas for Kahn's Theorem:
Randomly color edges; Uncolor incident edges with same color; iterate; FInish with a well chosen reserve of colors. □

Molly and Reed: $\chi'_\ell(G) = \Delta(G) + O\left(\Delta(G)^{1/2} \log^4 \Delta(G)\right)$

Kahn: Also holds for edge-coloring k-uniform linear hypergraphs. (A hypergraph is linear if any two vertices are contained in a most one hyperedge.)

1.2.3 Edge Coloring Multigraphs

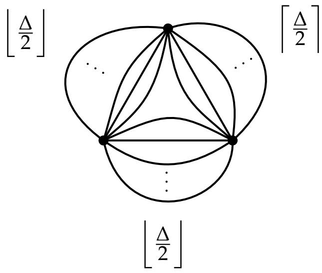
If G is a multigraph, let $\chi'(G)$ denote $\chi(L(G))$. Note that $L(G)$ is well-defined for multigraphs. Also note that we allow parallel edges but not loops, so that $L(G)$ is simple. We have a trivial bound:

$$\Delta(G) \le \omega(L(G)) \le \chi'(G) \le \Delta(L(G)) + 1 \le 2\Delta(G) - 1$$

Theorem 1.12: König, 1916

If G is a bipartite multigraph, then $\chi'(G) = \Delta(G)$.

Same proof works. However, Vizing's Theorem does not hold.



Theorem 1.13: Shannon, 1949

If G is a multigraph, then $\chi'(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor$.

We can't do better because it is tight for triangle.

multiplicity

The **multiplicity** of a multigraph G, denoted $\mu(G)$, is the maximum number of pairwise parallel edges in G.

Theorem 1.14: Vizing, 1964

If G is a multigraph, then $\chi'(G) \leq \Delta(G) + \mu(G)$.

Proof:

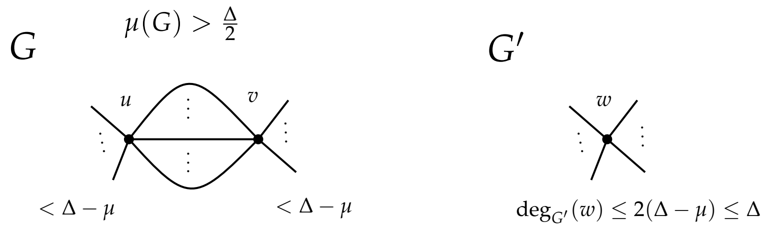
We carry over all definitions to multigraphs: Kempe chains, switching, missing colors, Vizing fan. Same proof works for disjoint missing color lemma in the context of multigraphs. Then we fix the finish as follows:

Let $k = \Delta(G) + \mu(G)$. Let $X := \bigcup_{i \in [n]} \phi(v_i)$. By lemma, we have $|X| \geq (k - \Delta(G))n = \mu(G)n$. There are at most $\mu(G)n - 1$ colored edges incident with v_0 and another vertex in T. So $\exists \alpha \in X$ such that $\nexists f = v_0 v_j$, $j \in [n]$ with $\phi(f) = \alpha$. Since $\alpha \notin \phi(v_0)$, $\exists e_{n+1} = v_0 v_{n+1}$ with $\phi(e_{n+1}) = \alpha$. But then $T + e_{n+1}$ is a larger Vizing fan with respect to e, v_0 and ϕ, contradicting the maximality of T. $\square$

Proof of Shannon's Theorem:

We proceed by induction. If $\mu(G) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor$, then desired outcome follows from Vizing's Theorem for multigraphs.

So we assume $\mu(G) > \frac{\Delta(G)}{2}$. Let $u, v \in V(G)$ such that there exist $\mu(G)$ parallel edges with ends u, v. Let G' be the multigraph obtained from G by identifying u and v to a new vertex w and deleting all loop incident with w.



Now $\deg_{G'}(w) \le 2(\Delta - \mu) \le \Delta$. Hence $\Delta(G') \le \Delta(G)$. By induction, G' has a $\left\lfloor \frac{3\Delta(G)}{2} \right\rfloor$-coloring ϕ. Extend ϕ to G as desired. Note this is possible as

$$\mu(G) + \deg_{G'}(w) \le 2\Delta - \mu(G) \le \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor$$

$\square$

Let

$$p(G) := \max \left\{ \left\lceil \frac{2|E(G[X])|}{|X| - 1} \right\rceil \;\middle|\; X \subseteq V(G) \right\}$$

Note that $\chi'(G) \ge p(G)$.

Goldberg 1979, indep. Seymour 1977

$\chi'(G) \le \max\{\Delta(G) + 1, p(G)\}$

Edmonds (matching polytope theorem) shows that the fractional chromatic index satisfies

$$\chi'_f(G) = \max\{\Delta(G) + 1, p(G)\}.$$

So the Goldberg-Seymour conjecture is equivalent to saying that $\chi'(G) = \chi'_f(G)$.

Kierstead Path

Suppose G is a graph, $e_1 = v_0v_1 \in E(G)$, S is a set of colors, and ϕ is an S-coloring of $G - e_1$. We say $T = (v_0e_1v_1e_2v_2 \dots e_nv_n)$ is a **Kierstead Path** with respect to the edge e_1 and the coloring ϕ if

- $v_0, v_1, v_2, \dots, v_n$ are all disjoint, and
- $\forall j, 1 \le j \le n$, $e_j = v_ju$ where $u \in \bigcup_{i<j}\{v_i\}$ and $u = v_{j-1}$, and
- $\forall j, 2 \le j \le n, \phi(e_j) \in \bigcup_{i<j} \phi(v_i)$.

Theorem (Kierstead)

The Disjoint missing colors lemma holds for Kierstead Paths.

Tashkinov Tree

Suppose G is a graph, $e_1 = v_0v_1 \in E(G)$, S is a set of colors, and ϕ is an S-coloring of $G - e_1$. We say $T = (v_0e_1v_1e_2v_2 \dots e_nv_n)$ is a **Tashkinov Tree** with respect to the edge e_1 and the coloring ϕ if

- $v_0, v_1, v_2, \dots, v_n$ are all disjoint, and
- $\forall j, 1 \le j \le n$, $e_j = v_ju$ where $u \in \bigcup_{i<j}\{v_i\}$, and
- $\forall j, 2 \le j \le n, \phi(e_j) \in \bigcup_{i<j} \phi(v_i)$.

Theorem (Tashkinov 2000)

The Disjoint missing colors lemma holds for Tashkinov Trees.

As time goes by, we have several upper bounds. The latest is

Theorem (Chen, Gao, Kim, Postle, Shan 2018)

$$\chi'(G) \leq \max\left\{\Delta(G) + \left\lceil (\Delta(G)/2)^{1/3} \right\rceil, p(G)\right\}$$

1.3 Thomassen's Theorem

What is the maximum list chromatic number of planar graphs? It is at least 4 since K_4 is planar, and at most 6 since planar graphs are 5-degenerate.

k-degenerate

A graph is ***k*-degenerate** if its vertices can be successively deleted so that when deleted, each has degree at most k. The degeneracy of a graph is the smallest k such that it is k-degenerate.

or

A k-degenerate graph is a graph in which every induced subgraph has a vertex with degree at most k.

Conjecture (Erdős, Rubin and Taylor 1979)

$\exists$ a planar graph with list chromatic number at least 5.

Theorem (Voigt 1993)

$\exists$ a planar graph with list chromatic number at least 5.

Conjecture (Erdős, Rubin and Taylor 1979)

Every planar graph has list chromatic number at most 5.

Theorem (Thomassen 1994)

Every planar graph has list chromatic number at most 5.

How to prove there exists a 5-list-coloring? Identification and Kempe chains do not work for list coloring, but we can prove something stronger.

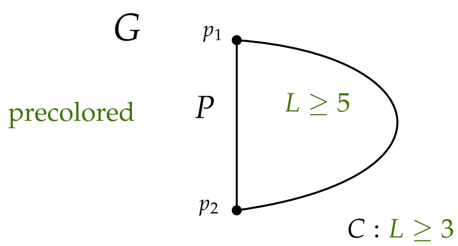
Theorem 1.15: Thomassen's Stronger Theorem

Let G be a connected plane graph, C be the boundary walk of the infinite face of G, and P be a path on at most two vertices in C.

If L is a list assignment of G such that

- $|L(p)| = 1\ \forall p \in V(P)$,
- $|L(v)| \geq 3\ \forall v \in V(C) \setminus V(P)$,
- $|L(w)| \geq 5\ \forall w \in V(G) \setminus V(C)$, and
- $G[V(P)]$ has an L-coloring,

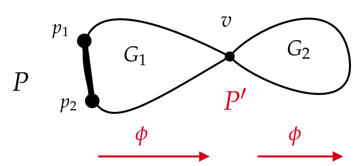
then G has an L-coloring.



Proof:
Let G be a counterexample with $|V(G)|$ minimized. Assume WLOG that $|V(P)| = 2$. Let $P = p_1p_2$.

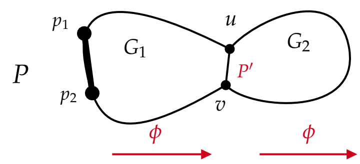
Global reduction: cutvertex/chord

Assume G has a cutvertex: $G_1 \cap G_2 = \{v\}, G_1 \cup G_2 = G$. Assume WLOG that $V(P) \subseteq V(G_1)$.



By minimality, G_1 has an L-coloring ϕ. By minimality, ϕ extends to an L-coloring of G_2.

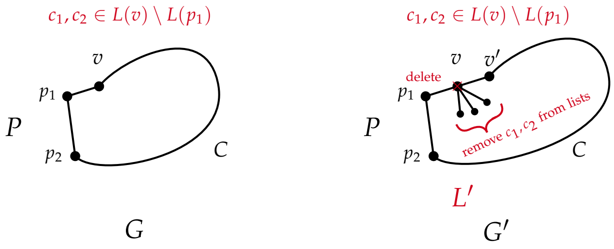
Assume G has a chord: $G_1 \cap G_2 = \{uv\}$, $G_1 \cup G_2 = G$. Assume WLOG $V(P) \subseteq V(G_1)$.



By minimality, G_1 has an L-coloring ϕ. By minimality, ϕ extends to an L-coloring of G_2.

Local Reduction

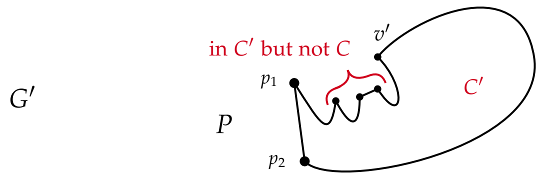
Let $v \neq p_2$ be neighbor of p_1 in C. Let $\{c_1, c_2\} \in L(v) \setminus L(p_1)$.



Let $G' = G - v$, let $v' \neq p_1$ be the neighbor of v in C and

$$L'(w) = \begin{cases} L(w) \setminus \{c_1, c_2\} & w \in N(v) \setminus \{v'\} \\ L(w) & \text{otherwise} \end{cases}$$

Let C' be the outer face of G'.



Since G is 2-connected, G' is connected. Note that $|L'(w)| \geq 3$ $\forall w \in V(C') \setminus V(P)$ because if $w \in N(v) \setminus \{p_1, v'\}$, then $w \in V(G) \setminus V(C)$ since C has no chord. $|L'(w)| \geq 5$ $\forall w \in V(G') \setminus V(C')$ since if $w \in N(v) \setminus V(C)$, then $w \in V(C')$. By minimality of G, $\exists L'$-coloring ϕ of G'. Now let $\phi(v) \in \{c_1, c_2\} \setminus \phi(v')$. $\square$

1.4 Coloring and List Coloring Planar Graphs

In 1852, Guthrie proposed Four Color Conjecture: Every planar graph is 4-colorable. It's equivalent formulation: Every bridgeless cubic planar graph is 3-edge colorable. The four color theorem was proved in 1976 by Kenneth Appel and Wolfgang Haken after many false proofs and counterexamples. Uses the method of discharging to show that every minimum counterexample contains one of 1834 unavoidable configurations. Show that each of these configurations is reducible (i.e. does not occur in a minimum counterexample). Second shorter proof by Robertson, Sanders, Seymour and Thomas in 1997 (only 633 configurations). Latter proof verified by formal proof software Coq in 2005. Interestingly, we can use fewer colors by excluding certain subgraphs.

Grötzsch's Theorem (1959) states that every triangle-free graph is 3-colorable. This does not extend to list coloring: Voigt (1995) states that $\exists$ a planar triangle-free graph that is not 3-list-colorable. In 1995, Thomassen's theorem states that every planar graph of girth at least five (i.e., no $\triangle$ or 4-cycles) is 3-colorable.

In 1976, Steinberg proposed a conjecture: Every planar graph without 4-cycles or 5-cycles is 3-colorable. Erdős, in 1991, proposed a question: What is the smallest k such that planar graphs with no cycles of length 4 to k are 3-colorable? Borodin, Glebov, Raspaud, Salavatipour (2005) showed $k = 7$ works. Cohen-Addad, Hebdige, Král', Li, Salgado (2016) showed Steinberg's conjecture is false. Dvořák, Postle, in 2018: planar and no 4 to 8 cycles is 3-list-colorable. But we have more questions on planar graphs:

- If a theorem guarantees one k-coloring, could we also prove a theorem guaranteeing many k-colorings?
- If a theorem guarantees one k-coloring, could we also prove a theorem guaranteeing a k-coloring if we add a few anomalies (e.g. precolored vertices, crossings, etc.)?

Not all planar graphs have (exponentially) many 4-colorings. Repeatedly adding new degree three vertices inside facial triangles of K_4 has only one 4-coloring up to permutation of colors. However, all planar graphs have (exponentially) many 5-colorings.

Proposition 1.16

Every graph has at least $2^{\frac{|V(G)|}{\chi(G)}}$ $(\chi(G)+1)$-colorings.

Proof:
Take the largest color class in a $\chi(G)$-coloring and recolor any subset with the $\chi(G)+1$-st color. □

These results can extend to list colorings. In 2007, Thomassen proved that If G is a planar graph and L is a 5-list-assignment of G, then G has at least $2^{\frac{|V(G)|}{9}}$ distinct L-colorings. In 1974, Aksenov proved that every planar graph with at most 3 triangles is 3-colorable. Havel's conjecture (1969) which states that $\exists D > 0$ s.t.: every planar graph where every pair of triangles is at distance $\geq D$ apart is 3-colorable, is proved by Dvořák, Král's and Thomas.

In 1997, Thomassen raised a question: Does $\exists D > 0$ s.t.: If G is a planar graph and X is a set of vertices $\geq D$ pairwise far apart, then every 5-coloring of X extends to G?

Proposition (Alberston 1998)

If G is a graph and X is a set of vertices ≥ 4 far apart, then every $\chi(G)+1$-coloring of X extends to G.

Proof:
Take a $\chi(G)$-coloring ϕ of G. Recolor every vertex x_i in X to its preferred color c_i. Recolor the neighbors of x_i colored c_i to color $\chi(G)+1$. This yields a $\chi(G)+1$-coloring of G, because $N(x_i)$ are disjoint/non-adjacent as distance ≥ 4. □

What about list coloring? By Thomassen's theorem: planar graphs with at most 2 precolored adjacent vertices are 5-list-colorable. What if the vertices are pairwise far apart? Alberston's Conjecture (1998): $\exists D > 0$ s.t.: every planar graph where every pair of precolored vertices is at distance $\geq D$ apart is 5-list-colorable, was proved by Dvořák et al. in 2017.

1.5 Discharging

Discharging is a counting method wherein:

- We assign charges to objects (e.g. vertices, edges, faces of a graph) such that the sum of the charges is negative (resp. positive)
- We redistribute the charge according to a set of discharging rules such that the sum is unchanged
- We derive a contradiction by showing the sum of the new charges is non-negative (resp. non-positive) given the assumed properties.

We use charges for plane graphs, natural choices come from Euler's formula. Recall Euler's formula:

Euler's formula

If G is a plane graph, then

$$|V(G)| - |E(G)| + |\mathscr{F}(G)| = 1 + |\mathscr{C}(G)|$$

where $\mathscr{F}(G)$ denote the set of faces of G and $\mathscr{C}(G)$ denotes the set of components of G.

1.5.1 Common Discharging Setups for Planar Graphs

Let G be a plane graph. Initial charges:

$$\begin{aligned} \mathrm{ch}_0(v) &= \deg(v) - 6 && \forall v \in V(G) \\ \mathrm{ch}_0(f) &= 2(|f| - 3) && \forall f \in \mathscr{F}(G) \end{aligned}$$

Total sum of charges:

$$\begin{aligned} &\sum_{v \in V(G)} \mathrm{ch}_0(v) + \sum_{f \in \mathscr{F}(G)} \mathrm{ch}_0(f) \\ &= \sum_{v \in V(G)} (\deg(v) - 6) + \sum_{f \in \mathscr{F}(G)} 2(|f| - 3) \\ &= \sum_{v \in V(G)} \deg(v) - 6|V(G)| + 2 \sum_{f \in \mathscr{F}(G)} |f| - 6|\mathscr{F}(G)| \\ &= 2|E(G)| - 6|V(G)| + 2(2|E(G)|) - 6|\mathscr{F}(G)| && \text{by handshaking} \\ &= 6(|E(G)| - |V(G)| - |\mathscr{F}(G)|) \\ &= -6(1 + |\mathscr{C}(G)|) && \text{by Euler's formula} \\ &= -12 && \text{if connected} \end{aligned}$$

1. Vertex-centric Setup
   - $\mathrm{ch}_0(v) = \deg(v) - 6\ \forall v \in V(G)$
   - $\mathrm{ch}_0(f) = 2(|f| - 3)\ \forall f \in \mathscr{F}(G)$
   - Sum: -12 if connected
   - Good for: unrestricted plane graphs.
2. Face-centric Setup
   - $\mathrm{ch}_0(v) = 2(\deg(v) - 3)\ \forall v \in V(G)$
   - $\mathrm{ch}_0(f) = |f| - 6\ \forall f \in \mathscr{F}(G)$
   - Sum: -12 if connected
   - Good for: cubic plane graphs
3. Balanced Setup
   - $\mathrm{ch}_0(v) = \deg(v) - 4\ \forall v \in V(G)$
   - $\mathrm{ch}_0(f) = |f| - 4\ \forall f \in \mathscr{F}(G)$
   - Sum: -8 if connected
   - Good for: triangle-free plane graphs or restrictions on triangles

A First Example

Some notation:

- k-vertex: degree is k
- k^+-vertex: degree is $\geq k$
- k^--vertex: degree is $\leq k$
- k-face, k^+-face, k^--face similar

Proposition 1.17

Every plane graph with $\delta(G) \geq 5$ contains a 5-vertex adjacent to a 7^--vertex.

Vertex-centric Setup: $\text{ch}_0(v) = \deg(v) - 6$, $\text{ch}_0(f) = 2(|f| - 3)$, Sum: ≤ -12.

Rule: Every 8^+-vertex sends $+\frac{1}{4}$ charge to each neighbor.

Let ch denote the final charge after applying rule.

Claim All final charges are nonnegative.

- Faces: $\text{ch}(f) = \text{ch}_0(f) \geq 0$
- 8^+-vertices:
$$\text{ch}(v) \geq \deg(v) - 6 - \frac{\deg(v)}{4} = \frac{3\deg(v) - 24}{4} \geq 0$$
since $\deg(v) \geq 8$.
- 6-vertex, 7-vertex: $\text{ch}(v) \geq \text{ch}_0(v) \geq 0$
- 5-vertex:
$$\text{ch}(v) = -1 + \frac{1}{4}(5) \geq \frac{1}{4} > 0$$
since every neighbor of v is an 8^+ vertex and sends $+\frac{1}{4}$ to v by Rule.

A Second Example

Proposition 1.18

Every plane graph with $\delta(G) \geq 3$ contains:

- a 3-vertex incident with a 5^--face, or
- a 5^--vertex incident with a 3-face.

Balanced Setup: $\text{ch}_0(v) = \deg(v) - 4$, $\text{ch}_0(f) = |f| - 4$, Sum: ≤ -8.

Rules:

1. Every 6^+-face sends $+\frac{1}{3}$ charge to each incident vertex.
2. Every 6^+-vertex sends $+\frac{1}{3}$ charge to each incident face.

Let ch denote the final charge after applying both rules.

Claim All final charges are nonnegative.

- 6^+-vertices:
$$\text{ch}(v) \geq \deg(v) - 4 - \frac{\deg(v)}{3} = \frac{2\deg(v) - 12}{3} \geq 0$$
since $\deg(v) \geq 6$.
- 4-vertex, 5-vertex: $\text{ch}(v) \geq \text{ch}_0(v) \geq 0$.
- 3-vertex:
$$\text{ch}(v) = -1 + \frac{1}{3}(3) = 0$$
since every face incident with v is a 6^+-face and sends $+\frac{1}{3}$ to v by Rule 1.
- Symmetric for faces.

A Final Example

Proposition 1.19

Every plane graph with $\delta(G) \geq 3$ contains:

- two adjacent 3-faces, or
- a j-face for some $4 \leq j \leq 9$, or
- a 10-face incident with only 3-vertices.

Here we use Face-centric Setup: $\mathrm{ch}_0(v) = 2(\deg(v) - 3), \mathrm{ch}_0(f) = |f| - 6$.

Rules:

1. Every 10^+-face sends $+1$ to each adjacent 3-face.
2. Every 4^+-vertex v sends $+1$ to each incident 10^+-face f where v is contained in a triangle sharing an edge with f.

Let ch denote the final charge after applying both rules.

Claim All final charges are nonnegative.

- 3-vertices: $\mathrm{ch}(v) = \mathrm{ch}_0(v) = 0$
- 4^+-vertex:
$$\mathrm{ch}(v) \geq 2(\deg(v) - 3) - \left\lfloor \frac{2\deg(v)}{3} \right\rfloor = \left\lceil \frac{4\deg(v)}{3} \right\rceil - 6 \geq 0$$
since $\deg(v) \geq 4$.
- 3-face:
$$\mathrm{ch}(f) = -3 + (+1)(3) = 0$$
since every face adjacent to f is a 10^+-face which sends $+1$ to f by Rule 1.
- 10^+-faces:
  - Loses 1 for every path along its boundary such that neighboring faces are triangles and ends are degree 3: Loses 1 to each triangle on the path by Rule 1, and Gains 1 for each interior vertex of path by Rule 2 (since these are 4^+-vertices)
  - Net loss is at most $\lfloor \frac{|f|}{2} \rfloor$
- 11^+-face:
$$\mathrm{ch}(f) \geq |f| - 6 - \left\lfloor \frac{|f|}{2} \right\rfloor = \left\lceil \frac{|f|}{2} \right\rceil - 6 \geq 0$$
- 10-face: if $\mathrm{ch}(f) < 0$, then 5 such paths, implies f incident to only 3-vertices, contradiction.

Corollary 1.20

Every planar graph with no 4 to 9-cycles is 3-colorable.

Proof:
By standard reduction, we may assume G is 2-connected and $\delta(G) \geq 3$. By proposition, $\exists$ a 10-face C incident with only 3-vertices. Delete $V(C)$, color by induction, extend to C. (Works since even cycles are 2-list-colorable) $\square$

1.6 Surfaces

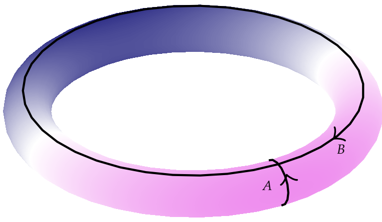
A surface is a closed, compact, connected, 2-dimensional manifold.

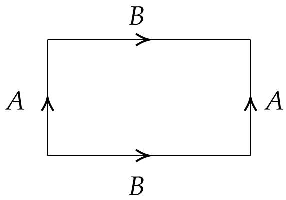
- closed: contains its boundary
- compact: every open cover has a finite subcover
- connected: there exists a path between any two points
- 2-dimensional manifold: each point has a neighborhood that is homeomorphic to $\mathbb{R}^2$.

Example:

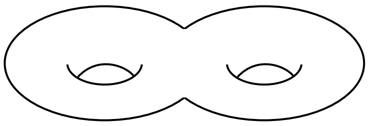
(2-dimensional) sphere

torus

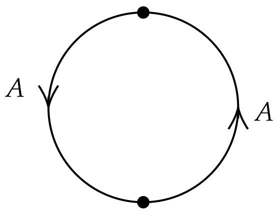




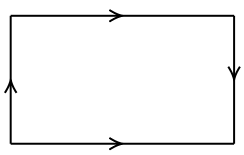
double torus



projective plane

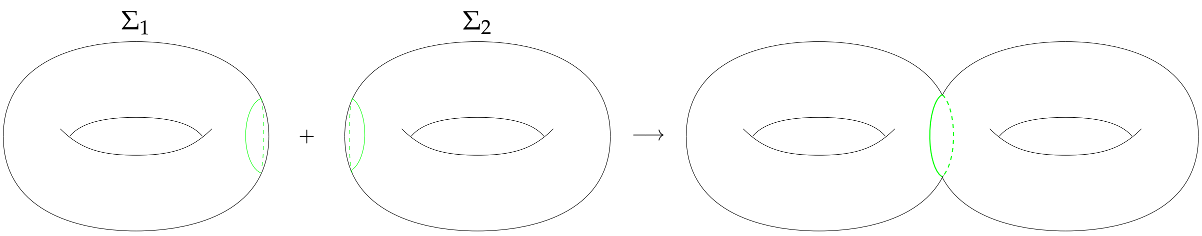


Klein bottle

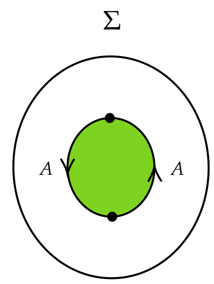


1.6.1 Operations

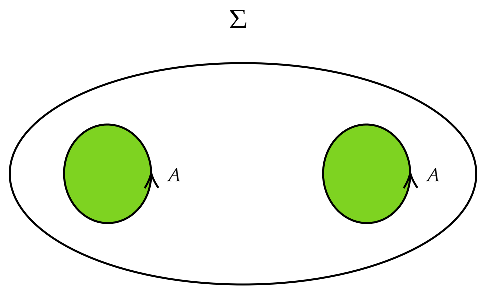
Connected Sum of two surfaces: remove a disk from both surfaces; glue along boundaries of disks.



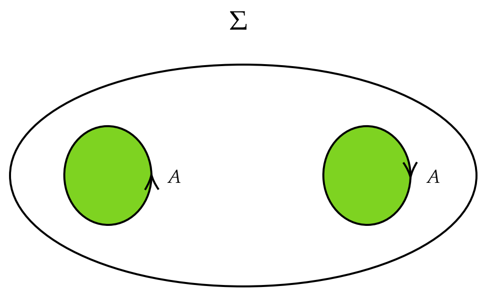
Adding a Crosscap to a surface: remove a disk and identify antipodal points of its boundary



Adding a Handle to a surface: remove two disks and identify their boundaries with same orientation



Adding a Twisted Handle to a surface: remove two disks and identify their boundaries with opposite orientation



Theorem 1.21: Classification to Surfaces Theorem

Every surface is homeomorphic to one of the following:

- S_h: adding $h \geq 0$ handles to a sphere
- N_k adding $k \geq 1$ crosscaps to a sphere

Equivalently:

- S_0 is the sphere,
- S_h (for $h \geq 1$) is the connected sum of h tori, and
- N_k is the connected sum of k projected planes.

A surface is **orientable** if there exists a consistent choice of orientation, **non-orientable** otherwise. Informally, a surface is orientable if "whenever you walk around left/right remain the same".

A closed curve γ in a surface is 2-sided if any small enough strip containing γ has 2 components after deleting γ. 1-sided if any small enough strip containing γ has only 1 component after deleting γ. Formally, A surface is orientable if and only if every closed curve is 2-sided.

A closed curve γ in a surface is **seperating** if the deleting of γ disconnects the surface. It is **contractible** if γ can be contracted to a point (or equivalently γ is 2-sided, separating, and at least one side bounds a disk).

One last operation (to reduce a surface): **Cutting along a Closed Curve**: Delete closed curve and add a disk to each side of the curve (two if 2-sided, one if 1-sided).

Proof Sketch for Classification:
Let Σ be a surface. If every closed curve in Σ is contractible, then Σ is a sphere. So WMA $\exists$ non-contractible closed curve γ in Σ. Consider cutting along γ. If γ is separating, then Σ is connected sum of two smaller surfaces. If 2-sided, non-separating, then Σ is obtained from a smaller surface by adding a handle. If 1-sided, then Σ is obtained from a smaller surface by adding a crosscap. $\square$

Note that adding a crosscap to S_h we get N_{2h+1}. Adding a handle to N_k we get N_{k+2}. The connected sum of S_h and N_k is N_{2h+k}. In all cases the result is non-orientable since the existence of a crosscap disorients the surface. The disorientation turns every handle into a twisted handle (which is equivalent to adding two crosscaps).

Euler's formula for Surfaces

The **Euler genus** $g(\Sigma)$ of a surface Σ is $g(\Sigma) := 2h + k$ where h is the number of handles and k is the number of crosscaps.

Euler's formula for Surfaces

If G is a graph embedded in a surface Σ, then

$$|V(G)| - |E(G)| + |\mathscr{F}(G)| = 1 + |\mathscr{C}(G)| - g(\Sigma)$$

where $\mathscr{F}(G)$ denote the set of faces of G and $\mathscr{C}(G)$ denotes the set of components of G.

Corollary 1.22

If G is a graph embedded in a surface Σ, then

$$|E(G)| \leq 3\big(|V(G)| - 2 + g(\Sigma)\big)$$

1.6.2 Coloring Graphs on Surfaces

For each surface Σ, what is the maximum chromatic number of graphs embeddable in Σ?

When Σ is the sphere, the answer is 4 by the Four Color Theorem. What about for surfaces of larger genus?

Theorem (Heawood 1890)

If G is a graph embedded in a surface Σ with $g(\Sigma) \geq 1$, then

$$\chi(G) \leq \frac{7 + \sqrt{1 + 24g(\Sigma)}}{2}$$

The **Heawood number** of a surface Σ:

$$H(\Sigma) := \left\lfloor \frac{7 + \sqrt{1 + 24g(\Sigma)}}{2} \right\rfloor$$

Note that $H(S_0) = 4, H(N_1) = 6, H(S_1) = H(N_2) = 7$

Lemma 1.23

If G is embeddable in a surface Σ with $g(\Sigma) \geq 1$, then $\delta(G) \leq H(\Sigma) - 1$.

Proof:
If $g = 1$, then $e \leq 3v - 3$ and hence $\delta \leq 5 = H(N_1) - 1$. So we assume $g \geq 2$. We assume $\delta \geq 6$, otherwise nothing to show. By Euler's formula and handshaking,

$$\delta v \leq 2e \leq 6v - 12 + 6g \implies (\delta - 6)v \leq 6g - 12$$

Using $v \geq \delta + 1$ and $\delta - 6 \geq 0$, we get

$$(\delta - 6)(\delta + 1) \leq 6g - 12 \implies \delta^2 - 5\delta + 6 - 6g \leq 0$$

□

By Lemma, it follows that a graph embeddable in Σ with $g \geq 1$ is $(H(\Sigma) - 1)$-degenerate, and hence $H(\Sigma)$-colorable by greedy.

Heawood's Conjecture (1890)

For all surfaces Σ with $g \geq 1$, $\exists$ a graph G embeddable in Σ with $\chi(G) = H(\Sigma)$.

Franklin (1930) showed it is false for Klein bottle ($\chi \leq 6$). Ringel and Youngs (1968) showed that it is true for all other Σ (they showed $K_{H(\Sigma)}$ embeds for all other Σ)

Modern Paradigm

Modern Paradigm (Thomassen 1990s): Perhaps "most graphs" embeddable on a surface have small chromatic number (independent of genus)?

Questions

For what k and Σ, are there only finitely many reasons that graphs embeddable in Σ are not k-colorable?

For what k and Σ, are "locally planar" graphs embedded in Σ k-colorable? We say that a graph G embedded in a surface S is locally planar if it does not contain short noncontractible cycles.

For what k and Σ, does exist a polynomial-time algorithm to decide if graphs embeddable in Σ are k-colorable?

k-critical

A graph G is ***k*-critical** if G has no $(k-1)$-coloring but very proper subgraph of G does.

Dirac (1952) and Albertson and Hutchinson (1978) showed: $K_{H(\Sigma)}$ is the only $H(\Sigma)$-critical graph embeddable in Σ.

For what k and Σ, do there exist only finitely many k-critical graphs embeddable in Σ?

Yes for $k \geq 8$ [Dirac, 1953], $k = 7$ [Gallai, 1963], $k = 6$ [Thomassen, 1997]. No for $k = 5$ and $\Sigma \neq S_0$ (Thomassen via construction of Fisk)

For what k and Σ does exist w such that all graphs embedded in Σ with edge-width $\geq w$ are k-colorable? **edge-width** of G, $\text{ew}(G)$: length of shortest non-contractible cycle.

Proposition 1.24

For a surface Σ, if finitely many k-critical, then locally planar $(k-1)$-colorable.

Proof:
Let L be list of k-critical graphs embedded in Σ. Let $w = \max\{\mathrm{ew}(G) : G \in L\} + 1$. If $\mathrm{ew}(G) \geq w$, then $\nexists H \in L$ with $H \subseteq G$. Hence G is $(k-1)$-colorable. □

"Locally planar" graphs in Σ are 5-colorable (by Thomassen). Thomassen (1993) showed $\mathrm{ew} \geq 2^{\Omega(g)}$ suffices. Postle, Thomas (2018) showed $\mathrm{ew}(G) \geq \Omega(\log g)$ suffices. For $\Sigma \neq S_0$, there exist graphs of arbitrarily large edge-width that are not 4-colorable (Thomassen).

For what k and Σ, does exist a polynomial-time algorithm to decide if graphs embeddable in Σ are k-colorable?

Proposition 1.25

For a surface Σ: if finitely many k-critical, then there exists polytime algorithm to decide $(k-1)$-colorable.

Proof:
Let L be list fo k-critical graphs embedded in Σ. For all $H \in L$: tet if $H \subseteq G$. If yes for some H, then return NO; otherwise return YES.

This runs in $|V(G)|^{\max\{|V(H)|:H \in L\}}$ time.

Actually subgraph testing is linear-time on a fixed surface. □

For a fixed surface Σ: there exists a linear-time algorithm to decide if a graph embedded in Σ is 5-colorable (by Thomassen). In 2013, Dvořák, Kawarabayashi found that there exists $|V(G)|^{O(g)}$ algorithm to find a 5-coloring if it exists. Postle (2019) found that there exists a linear-time algorithm to find a 5-coloring if it exists.

For $\Sigma \neq S_0$, it is an open problem whether there exists a poly-time algorithm to decide if a graph embedded in Σ is 4-colorable.

What about graphs with larger girth (triangle-free, girth ≥ 5)?

- Finitely many k-critical graphs:

  Yes for $k \geq 5$, triangle-free (fairly easy).

  No for $k = 4$, triangle-free, $\Sigma \neq S_0$ (Thomassen).

  Yes for $k = 4$, girth ≥ 5 (Thomassen 2003).

- Triangle-free:

  Locally planar triangle-free graphs are 3-colorable if Σ is orientable (Dvořák, Král' and Thomas 2008-2020+). There exists a polytime algorithm to decide 3-colorable for triangle-free graphs on any Σ (Dvořák, Král' and Thomas 2009)

What about list-coloring?

k-list-critical

A graph G is ***k*-list-critical** if ther exists a $(k-1)$-list-assignment L such that G hsa no L-coloring but every proper subgraph of G does.

Finitely many k-list-critical graphs: Yes for $k \geq 7$, $k = 6$, $k = 5$ triangle-free, $k = 4$, girth ≥ 5.

2

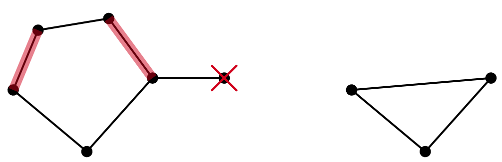
Graph minors

2.1 Minors

minor

We say that a graph G has an H **minor** if a graph is isomorphic to H can be obtained from a subgraph of G by contracting edges.

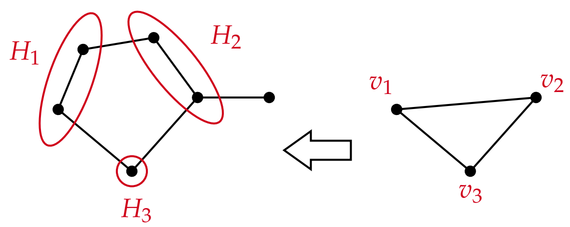
Example:



model

Let H be a graph with $V(H) = \{v_1, \ldots, v_t\}$. A **model of** H in a graph G is a collection of vertex-disjoint connected subgraphs $H_1, \ldots, H_t$ such that $\forall i \neq j \in [t]$ with $v_i v_j \in E(H)$, H_i is adjacent to H_j (i.e., there exists an edge with one end in H_i and the other end in H_j).

Example:



It is not hard to see that G has an H minor if and only if there exists a model of H in G.

subdivision

We say that a graph G has a **subdivision of** H (aka **topological minor**) if a graph isomorphic to H can be obtained fro ma subgraph of G by suppressing vertices of degree two.

Equivalently, there exists

- an ordering $v_1, \ldots, v_{|V(H)|}$ of H,
- distinct vertices $u_1, \ldots, u_{|V(H)|}$ of G and
- a collection of paths $\mathscr{P} = \{P_{ij} : v_i v_j \in E(H)\}$ where P_{ij} has ends u_i and u_j and $V(P_{ij}) \cap \{u_1, \ldots, u_{|V(H)|}\} = \{u_i, u_j\}$ for all $i \neq j \in [|V(H)|]$.

Kuratowski's Theorem (1930)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a topological minor.

Kuratowski's Theorem (Equivalent formulation, Wagner 1937)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

We say a class of graphs $\mathscr{G}$ is **minor-closed** if $\forall G \in \mathscr{G}$ and minor H of G, we have that $H \in \mathscr{G}$. The minor-minimal graphs not in $\mathscr{G}$ are called the **forbidden minors** of G (aka **Kuratowski set**).

For what minor-closed $\mathscr{G}$ is the Kuratowski set finite?

What is the structure of graphs with no K_t minor?

Some Minor-Closed Classes

Forests. Forbidden minors: K_3.

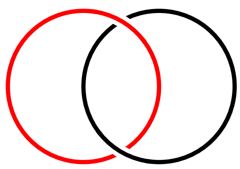
Outerplanar graphs. A graph is **outerplanar** if it can be embedded in the plane with all vertices incident to the infinite face. Forbidden minors: $K_{2,3}$ and K_4.

Projective Planar graphs. Forbidden Minors: 35 (Archdeaon 1981)

Graphs embeddable in a fixed surface Σ. Forbidden Minors: ??

Linkless graphs. A graph G is **linkless** if G can be embedded in $\mathbb{R}^3$ such that no two vertex-disjoint cycles of G are "linked". Forbidden Minors (Robertson, Seymour and Thomas 1995): The Petersen family (7 graphs) which includes the Petersen graph and K_6.

linked



Struture for forbidding small complete minors:

- No K_1 minor: empty.
- No K_2 minor: independent set.
- No K_3 minor: forests.

What is the structure of K_4-minor-free graphs?

k-sum

A k**-sum** of two graphs G_1 and G_2: identify vertices of a K_k in each graph and possibly delete the overlapping edges.

- No K_2 minor: 0-sum of copies of K_1.
- No K_3 minor: subgraph of ≤ 1-sum of copies of K_2.

Theorem 2.1

If no K_4 minor, then has subgraph of ≤ 2-sum of copies of K_3.

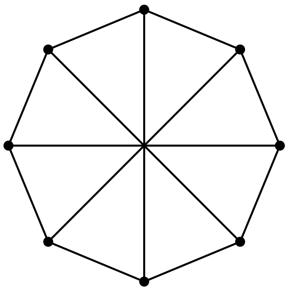
Proof:
By induction. If G has/is

- a ≤ 1-cut, then G is ≤ 1-sum of G_1, G_2.
- a 2-cut $\{u, v\}$, then G is the 2-sum of $G_1 + uv$, $G_2 + uv$, both of which are minors of G.
- 3-connected, then G contains a K_4 minor (i.e., an induced cycle C and 3 paths from a vertex v not in C to C).

□

What is the structure of K_5-minor-free graphs?

Much more complicated, because it includes all planar graphs. But that's not all, because it includes $K_{3,3}$ and its supergraph V_8 (the Möbius ladder on 8 vertices):



V_8

Theorem (Wagner 1937)

G has no K_5-minor if and only if G can be obtained by ≤ 3-sums of planar graphs and V_8.

2.2 Well-Quasi-Ordering

A **quasi-ordering** is a relation that is both reflexive ($x \leq x$) and transitive ($x \leq y, y \leq z$ implies $x \leq z$).

well-quasi-ordering

A **well-quasi-ordering** is a quasi-ordering $\leq$ such that for every infinite sequence $x_0, x_1, \ldots$ there exists $i < j$ such that $x_i \leq x_j$.

We call (x_i, x_j) a good pair. We say an infinite sequence is good if it has a good pair, bad otherwise.

We have some basic facts about WQO:

Proposition

A quasi-ordering $\leq$ on X is a well-quasi-ordering if and only if X contains neither an infinite antichain nor a strictly decreasing sequence $x_0 > x_1 > \ldots$

Corollary

If X is well-quasi-ordered by $\leq$, then every infinite sequence in X has an infinite increasing subsequence.

The minor relation is a quasi-ordering on graphs. Wagner's conjecture (1960s) was then presented as the theorem:

The Graph Minor Theorem (Robertson and Seymour 1986-2004)

The minor relation is a well-quasi-ordering.

Proof:
Over 20 papers and hundereds of pages! □

Corollary

The Kuratwoski set for any minor-closed property is finite.

Theorem (Robertson and Seymour)

There exists a cubic-time algorithm to test if a graph contains a fixed graph H as a minor.

A graph G is **knotless** if G can be embedded in $\mathbb{R}^3$ such that no cycle is "knotted".

Before the graph minor theorem, it was open whether knotlessness is decidable, i.e. whether any algorithm exists to decide if a graph is knotless. By the graph minor theorem, the Kuratowski set for knotlessness is finite and hence there exists a cubic-time algorithm to decide if a graph is knotless.

Proposition 2.2

If X is well-quasi-ordered by $\leq$, then so is $|X|^{<\omega}$.

Proof:
Suppose not. Choose $A_0, A_1, \ldots, A_n, \ldots$ iteratively where the sequence extends to a bad seqeunce, and $|A_n|$ is minimized for each n. For all n, pick $a_n \in A_n$ and let $B_n = A_n \setminus \{a_n\}$. Since X is well-quasi-ordered, by corollary the sequence $(a_n)_{n\in\mathbb{N}}$ has an infinite increasing subsequence $(a_{n_i})_{i\in\mathbb{N}}$. By minimality of A_{n_0}, the sequence $A_0, \ldots, A_{n_0-1}, B_{n_0}, B_{n_1}, B_{n_2}, \ldots$ is good. Since $(A_n)_{n\in\mathbb{N}}$ is nad, the good pair is not of the form (A_i, A_j) or (A_i, B_j). Hence it is of the form (B_i, B_j). But then $(B_i \cup \{a_i\} = A_i), B_j \cup \{a_j\} = A_j$ is a good pair, contradiction. □

Theorem 2.3: Kruskal 1960

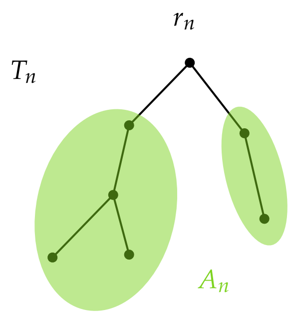
Trees are well-quasi-ordered by the topological minor relation.

More strongly, they are well-quasi-ordered in the rooted sense: Let T, T' be rooted trees with roots r, r'. We write $T \leq T'$ if there exists an isomorphic ϕ from a subdivision T to a subtree of T' that preserves the tree-order on $V(T)$ associated with T and r. (i.e., if x is a parent of y in T, then $\phi(x)$ is a parent of $\phi(y)$ in T').

Theorem 2.4

Rooted trees are well-quasi-ordered by $\leq$.

Proof of Kruskal's:
Suppose not. Choose $T_0, T_1, \ldots, T_n, \ldots$ iteratively where the sequence extends to a bad seqeunce, and $|V(T_n)|$ is minimized for each n. For all n, let r_n be the root of T_n and let A_n be the set of components of $T_n - r_n$ whose neighbors of r_n.



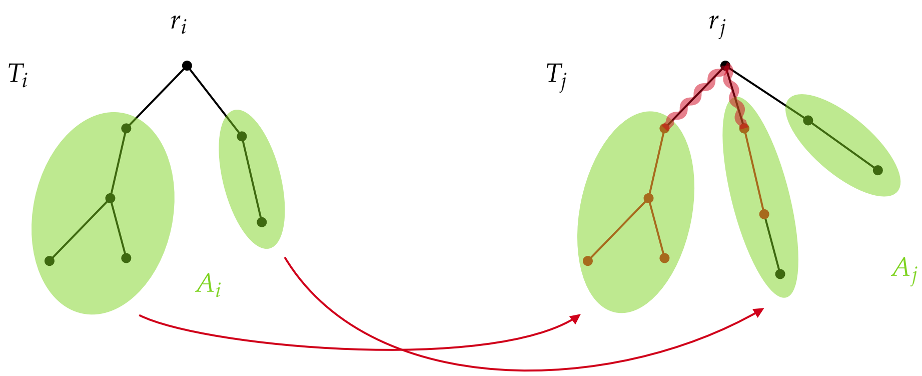
Claim $A := \bigcup_{n\in\mathbb{N}} A_n$ is well-quasi-ordered.

Proof:
Let $(T^k)_{k\in\mathbb{N}}$ be any sequence of trees in A. For all k, let $n(k)$ such that $T^k \in A_{n(k)}$. Pick k with smallest $n(k)$. By the minimality of $n(k)$, the sequence

$$T_0, \ldots, T_{n(k)-1}, T^k, T^{k+1}, T^{k+2}, \ldots$$

is good. A good pair (T, T') of the sequence above has no member in $T_0, \ldots, T_{n(k)-1}$ an hence the sequence $(T^k)_{k\in\mathbb{N}}$ is good. □

By Subset Lemma, $[A]^{<\omega}$ is well-quasi-ordered. Hence the sequence $(A_n)_{n\in\mathbb{N}}$ has a good pair (A_i, A_j).



But then (T_i, T_j) is a good pair, a contradiction, where we map $\phi(r_i)$ to r_j and add paths from r_j to the images in T_j of the roots of A_i. □

2.3 Tree-Decompositions and Tree-Width

How 'tree-like' is a graph?

tree-decomposition

A **tree-decomposition** of a graph G is a pair $(T, \mathscr{V})$ where

- T is a tree,
- $\mathscr{V} = (V_t)_{t \in V(T)}$ is a family of vertex sets V_t of G (called the **bags** of decomposition)

that satisfy:

(T1) $V(G) = \bigcup_{t \in T} V_t$, and

(T2) $\forall e \in E(G), \exists t \in V(T)$ such that both ends of e lie in V_t, and

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 is on the path in T connecting t_1 and t_3.

tree-decomposition (equivalent definition)

A **tree-decomposition** of a graph G is a pair $(T, \mathscr{V})$ where

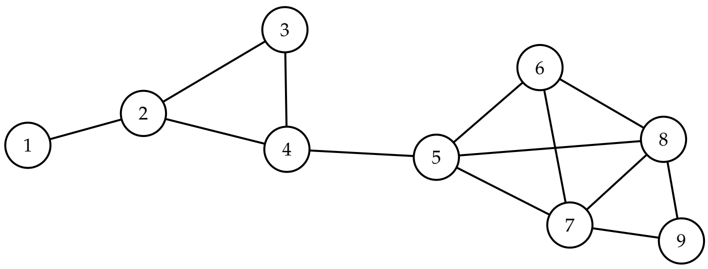
- T is a tree,
- $\mathscr{V} = (V_t)_{t \in V(T)}$ is a family of vertex sets V_t of G (called the **bags** of decomposition)

that satisfy:

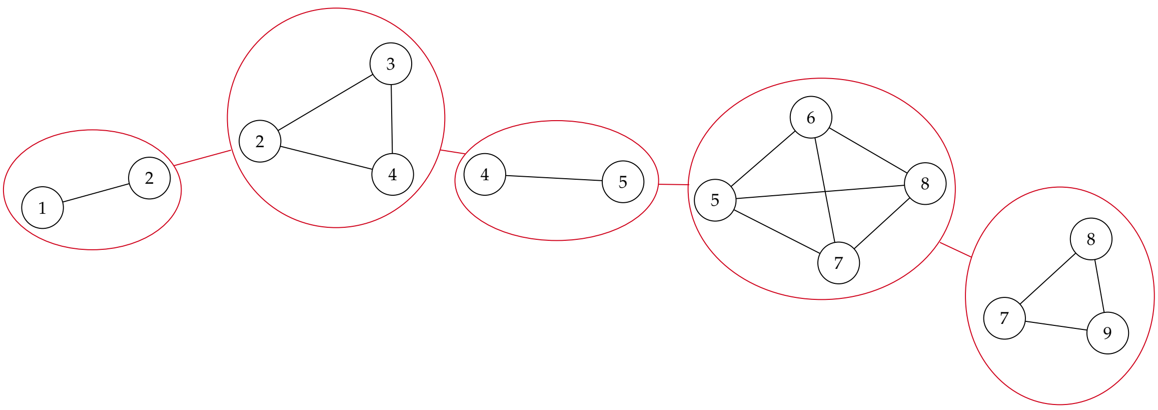
(T1') $\forall v \in V(G)$, the set $T_v = \{t \in T : v \in V_t\}$ is a nonempty subtree of T, and

(T2') $\forall e = uv \in E(G), T_u \cap T_v \neq \varnothing$.

Example:

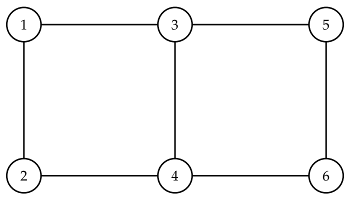


Tree-Decomposition:

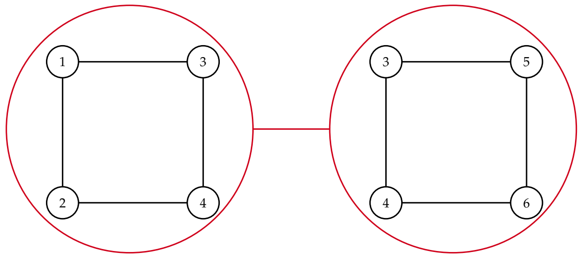


We call the subgraphs $G[V_t]$ the **torsos** of the composition.

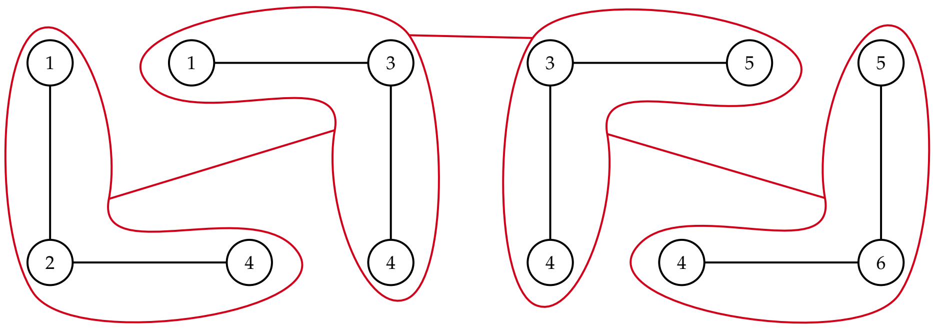
Example:



Tree-Decomposition:



Can we do "better"?



Proposition 2.5

If $(T, \mathscr{V})$ is a tree-decomposition of a graph G and H is a subgraph of G, then

$$(T, \mathscr{V}') = \left(V_t \cap V(H)\right)_{t \in T}$$

is a tree-decomposition of H.

Proof:
Since if (T1′), (T2′) hold for $(T, \mathscr{V})$, then they hold for $(T, \mathscr{V}')$, then it follows from alternative definition, $(T, \mathscr{V}')$ is a tree-decomposition of H. $\square$

Proposition 2.6

If $(T, \mathscr{V})$ is a tree-decomposition of a graph G and $\mathscr{H} = (H_1, \ldots, H_t)$ is a model of a graph H in G, then

$$(T, \mathscr{V}') = \left(\{v_i \in V(H) : V_t \cap V(H_i) \neq \varnothing\}\right)_{t \in T}$$

is a tree-decomposition of H.

Proof:
By (T1′) and (T2′), it follows that $\{v_i \in V(H) : V_t \cap V(H_i) \neq \varnothing\}$ is a nonempty subtree of T. Then (T2′) holds for $(T, \mathscr{V}')$ since it held for $(T, \mathscr{V})$. $\square$

width

The **width** of a tree-decomposition of $(T, \mathcal{V})$ of a graph G is defined to be

$$\max\{|V_t| - 1 : t \in T\}$$

tree-width

The **tree-width** of a graph G, denoted $\text{tw}(G)$, is the minimum width over all tree-decompositions of G.

Note the '-1' is just so that trees have tree-width 1.

Proposition 2.7

If H is a minor of G, then $\text{tw}(H) \leq \text{tw}(G)$.

Proof:
Follows from Subgraph/Minor Propositions. □

$\text{tw}(K_n) = n - 1$

Proposition 2.8

If G is the k-sum of G_1 and G_2, then

$$\text{tw}(G) \leq \max\{\text{tw}(G_1), \text{tw}(G_2)\}$$

Proof:
Put an edge between optimal tree-decompositions of G_1 and G_2 together at bags containing the common K_k vertices. □

Proposition 2.9

$\text{tw}(G) \leq 1$ if and only if G is a forest.

Proof:
The forward direction follows since $\text{tw}(K_3) = 2$.

The backward direction follows since a forest is a subgraph of $\leq$ 1-sum of K_2's. □

Proposition 2.10

$\text{tw}(G) \leq 2$ if and only if G has no K_4 minor.

Proof:
The forward direction follows since $\text{tw}(K_4) = 3$.

The backward direction follows since a K_4-minor-free graph is a subgraph of $\leq$ 2-sum of K_3's. □

Theorem (Robertson and Seymour 1990)

For all $k > 0$, the graphs of tree-width $< k$ are well-quasi-ordered by the minor relation.

For $k = 1$, just independent sets. For $k = 2$, this is Kruskal's theorem. For $k \geq 3$, similar to proof of Kruskal's theorem except we have to iteration $\text{tw}(G) - 1$ times.

Many NP-hard problems can be solved in polynomial time on graphs of bounded tree-width via dynamic programming. For example, deciding k-coloring:

- Root a tree-decomposition $(T, \mathscr{V})$ of width $\text{tw}(G)$ at some vertex r
- Iteratively compute $\Phi(t)$ staring from the leaves
- where $\Phi(t)$ is the set of k-colorings of $G[V_t]$ that extend toa k-coloring of the subgraph induced by $\{V_{t'} : t' \text{ is a child of } t\}$
- Return YES if and only if $\phi(r) \neq \varnothing$.

2.4 Brambles and Grids

When does a graph have large tree-width?

touch

Let G be a graph. Two subsets $A, B \subseteq V(G)$ **touch** if $A \cap B \neq \varnothing$ or $\exists$ and edge of G with one end in A and the other end in B.

bramble

A **bramble** $\mathscr{B}$ in a graph G is a collection of connected subsets of $V(G)$ that are pairwise touching.

cover of $\mathscr{B}$

A **cover of** $\mathscr{B}$ is a subset $X \subseteq V(G)$ such that $X \cap B \neq \varnothing$ for all $B \in \mathscr{B}$.

order of $\mathscr{B}$

The **order of** $\mathscr{B}$ is the minimum size of a cover of $\mathscr{B}$.

Theorem 2.11: Treewidth Duality Theorem (Seymour and Thomas, 1993)

A graph has tree-width $< k$ if and only if it contains no bramble of order $> k$.

Proof of forward direction:

Let $(T, \mathscr{V})$ be a tree-decomposition of G of width $< k$ and let $\mathscr{B}$ be a bramble. We claim that there exists a bag that is a cover of $\mathscr{B}$.

Proof:

Suppose not. Orient every edge $e = t_1t_2$ of T as follows:

- Since $V_{t_1} \cap V_{t_2}$ does not cover $\mathscr{B}$, there exist sets in $\mathscr{B}$ disjoint from $V_{t_1} \cap V_{t_2}$.
- Since all such sets mutually touch, there exists a unique component of $T - e$ containing such sets.
- Orient e toward that component.

Since T is a tree, there exists a sink t of the orientation. Now V_t covers $\mathscr{B}$, contradiction. $\square$

By Claim, every bramble has order $\leq k$ (the max size of a bag). □

Which graphs have large brambles (and hence large tree-width)?

K_n has a bramble of order k (namely $\mathscr{B} = \{\{v\} : v \in V(G)\}$). What other graphs?

grid

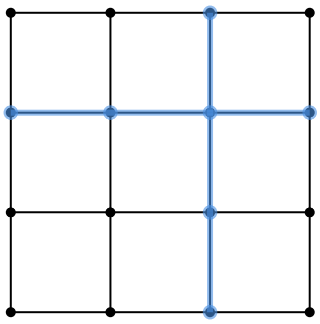
The $k \times k$ **grid** is the graph with vertices $[k]^2$ and edge set

$$\{(i,j)\,(i',j') : |i-i'| + |j-j'| = 1\}$$

crosses

The **crosses** of the grid are the k^2 sets

$$C_{ij} = \{(i,\ell) : \ell \in [k]\} \cup \{(\ell,j), \ell \in [k]\}$$



Then $\mathscr{B} = \{C_{i,j} : i,j \in [k]\}$ is a bramble of order k: The C_{ij} are connected and mutually touch; any row or column covers $\mathscr{B}$; any set of size $< k$ misses both a row and column and hence fails to cover $\mathscr{B}$.

Grid Theorem (Robertson and Seymour 1986)

There exists $f(k)$ such tbat every graph of tree-width at least $f(k)$ contains a $k \times k$ grid as a minor.

The latest result (Chuzhoy and Tan 2019) shows that $\Omega(k^9)$ suffices.

Corollary 2.12

Let H be a graph. Then the graphs without H as minor have bounded treewidth if and only if H is planar.

Proof:
If H is nonplanar, then the class of H-minor-free graphs contains all planar graphs, hence all grids, and hence has unbounded treewidth.

So suppose H is planar. Then H is a minor of some grid. Because we can embed H in plane, and take superfine grid and 'snap' H to grid. By the Grid Theorem, the class of H-minor-free graphs has bounded tree-width. □

The Treewidth Duality Theorem is related to the following game:

Cops and Robbers

Let G be a graph. There are k cops each of whom at any time are standing on a vertex of G, or are in a helicopter (i.e., removed from the game). A robber who is infinitely fast and can run at any time, cannot run through a cop, is always visible to the cops.

The cops win if they land a helicopter on the vertex occupied by the robber. The robber wins if it eludes capture.

Robber Winning Strategy

A winning robber strategy is (essentially) equivalent to:

haven

Let G be a graph. A **haven** in G of order $k+1$ is a function β which assigns to each subset X of $V(G)$ of size $\leq k$ a component $G-X$ such that $\beta(X)$ touches $\beta(Y)$ for all $X, Y \in [V(G)]^{\leq k}$.

Proposition 2.13

A graph G has a bramble of order k if and only if it has a haven of order k.

Proof:

Suppose $\mathscr{B}$ is a bramble of order k. For each $X \in [V(G)]^{<k}$, there exists $B \in \mathscr{B}$ such that $B \cap X = \varnothing$. Let $\beta(X)$ be the component of $G-X$ containing H. Then β is a haven of order k.

Then we suppose β is a haven of order k. Let $\mathscr{B} = \{\beta(X) : X \in [V(G)]^{<k}\}$. Then $\mathscr{B}$ is a bramble of order k. □

Winning Cop Strategy

A winning cop strategy is (essentially) a search tree T where: each vertex $t \in V(T)$ has the position V_t of the cops, and the robber is in some component of $G - V_t$. The tree branches then depending on how the robber moves in response to the cops.

monotonic winning cop strategy

A winning cop strategy is **monotonic** if whenever a cop leaves a vertex, then no cop ever returns to that vertex.

What is this equivalent to? Namely that $T_v = \{t \in T : v \in V_t\}$ is connected. Note that in order for the cops to win: every vertex needs to be occupied by a cop at some time, and every edge needs to have both cops on its ends at some time.

Proposition 2.14

A monotonic winning cop strategy is equivalent to a tree-decomposition of width $< k$.

Tree-width Duality Theorem (Equivalent Formulation)

k cops have a monotonic winning strategy if and only if they have a winning strategy.

Now let's prove the backward direction of Treewidth Duality Theorem. Let's prove some preliminary lemmas first.

Lemma 2.15

Let $(T, \mathscr{V})$ be a tree-decomposition of a graph G. If $e \in E(T)$ and T_1, T_2 are the components of $T - e$, then $V_{t_1} \cap V_{t_2}$ separates $\bigcup_{t \in T_1} V_t$ from $\bigcup_{t \in T_2} V_t$.

Proof:
Suppose not. That is, there exists $v_1 v_2 \in E(G)$ with $v_1 \in \bigcup_{t \in T_1} V_t \setminus V_{t_2}$ and $v_2 \in \bigcup_{t \in T_2} V_t \setminus V_{t_1}$. By axiom (T2'), there exists $t_0 \in T$ with $\{v_1, v_2\} \subseteq V_{t_0}$. WLOG assume $t_0 \in T_1$. By axiom (T1'), T_{v_2} is a subtree. Since $v_2 \in V_{t_0} \cap \bigcup_{t \in T_2} V_t$, we find that $v_2 \in V_{t_1} \cap V_{t_2}$, a contradiction. $\square$

Lemma 2.16

Any set of vertices separating two covers of a bramble also covers that bramble.

Proof:
Since each set in the bramble is connected and meets both covers, it also meets any set separating those covers. $\square$

B-admissible tree-decomposition

Let G be a graph with no bramble of order $> k$. If $\mathscr{B}$ is a bramble of G, then a **$\mathscr{B}$-admissible tree-decomposition** is one where any bag of size $> k$ fails to cover $\mathscr{B}$.

Lemma 2.17: Key Lemma

Let G be a graph with no bramble of order $> k$. For every bramble $\mathscr{B}$ of G, there exists a $\mathscr{B}$-admissible tree-decomposition of G.

When $\mathscr{B} = \varnothing$, a $\mathscr{B}$-admissible tree-decomposition is simply one of width $< k$, since every set covers the empty bramble. Hence the Key Lemma implies the hard direction of the Tree-width Duality Theorem.

Proof of the Key Lemma:
Suppose not. Let $\mathscr{B}$ be a bramble of G such that there does not exist a $\mathscr{B}$-admissible tree-decomposition of G, and subject to that, $|\mathscr{B}|$ is maximized. Such $\mathscr{B}$ exists since every bramble has at most $2^{|V(G)|}$ sets.

Let $X \subseteq V(G)$ be a cover of $\mathscr{B}$ with $|X|$ minimized. Note $\ell := |X|$ is the order of $\mathscr{B}$ and hence $\ell \leq k$. If $X = V(G)$, then the tree-decomposition with X as its only bag is $\mathscr{B}$-admissible, a contradiction. So we assume $X \neq V(G)$.

Key Claim For all component C of $G - X$, there exists a $\mathscr{B}$-admissible tree-decomposition of $G[X \cup V(C)]$ with X as a bag.

The Claim implies the Key Lemma as follows: Identify the tree-decompositions for all the components of $G - X$ at the bags equal to X. This is a tree-decomposition of G (as it satisfies the axioms) and $\mathscr{B}$-admissible (since every bag of size $> k$ fails to vover $\mathscr{B}$), a contradiction.

Now we prove the key claim.

Let C be a fixed component of $G - X$, $H := G[X \cup V(C)]$, $\mathscr{B}' = \mathscr{B} \cup \{C\}$.

Case 1: $\mathscr{B}'$ is not a bramble. Then C fails to touch some element of $\mathscr{B}$, hence $Y := V(C) \cup N(C)$ fails to cover $\mathscr{B}$, hence the tree-decomposition of H with bags X and Y is $\mathscr{B}$-admissible as desired.

Case 2: $\mathscr{B}'$ is a bramble. Note that $C \notin \mathscr{B}$ since X covers $\mathscr{B}$ and $C \cap X = \varnothing$. Hence $|\mathscr{B}'| > |\mathscr{B}|$. By maximality of $\mathscr{B}$, there exists a $\mathscr{B}'$-admissible tree-decomposition $(T, \mathscr{V})$ of G. Since $(T, \mathscr{V})$ is not $\mathscr{B}$-admissible, there exists a bag V_s of size $> k$ that covers $\mathscr{B}$.

By Lemma 2.16, every set separating X and V_s also covers $\mathscr{B}$. Then since X is a minimum cover of $\mathscr{B}$, every set separating X and V_s has size at least ℓ. By Menger's Theorem, there exist ℓ vertex-disjoint paths $P_1, \ldots, P_\ell$ from X to V_s. Since $(T, \mathscr{V})$ is $\mathscr{B}'$-admissible, then $V_s \cap C = \varnothing$. Hence $P_i \cap V(H) = \{x_i\}$ for all $i \in [\ell]$.

For all $i \in [\ell]$, pick a t_i with $x_i \in V_{t_i}$. Let Q_i be the path in T from s to t_i. We construct a new tree-decomposition $(T, \mathscr{W})$ of H where for each $t \in T$, we let

$$W_t := \big(V_t \cap V(H)\big) \cup \big\{x_i : t \in V\left(Q_i\right)\big\}$$

This is a tree-decomposition of H since it is the tree-decomposition of H induced by $(T, \mathscr{V})$ except with some extra vertices in bags, yet T_{x_i} is still a subtree since $x_i \in V_{t_i}$.

Note that $W_s = X$ is a bag of $(T, \mathscr{W})$. We will show that $(T, \mathscr{W})$ is a $\mathscr{B}$-admissible as desired.

Subclaim 1 $|W_t| \le |V_t|$ for all $t \in V(T)$.

Proof:

Let $I_t = \{i \in [\ell] : x_i \in W_t \setminus V_t\}$. It follows from Lemma 2.15 that $V(P_i) \cap V_t \neq \varnothing$ for all $i \in I_t$. Since $P_i \cap (V(H) \setminus X) = \varnothing$, we find that

$$|W_t| \le |V_t| - |I_t| + \sum_{i \in I} |V\left(P_i\right) \cap V_t| \le |V_t|$$

as claimed. □

Subclaim 2 $(T, \mathscr{W})$ is $\mathscr{B}$-admissible.

Proof:

Suppose not. That is, there exists $t \in T$ such that $|W_t| > k$ and W_t covers $\mathscr{B}$. By Subclaim 1, $|V_t| \ge |W_t| > k$. Since $(T, \mathscr{V})$ is $\mathscr{B}$-admissible, V_t fails to cover $\mathscr{B}$. That is, there exists $B \in \mathscr{B}$ such that $V_t \cap B = \varnothing$. Yet $W_t \cap B \neq \varnothing$. Hence there exists $i \in [\ell]$ with $x_i \in W_t \setminus V_t$ and $x_i \in B$. Since B is a connected set meeting both V_{t_i} and V_s, it follows from Lemma 2.15 that $B \cap V_t \neq \varnothing$, a contradiction. □

□

2.5 The Erdős-Pósa Theorem

Theorem 2.18

Let T be a tree. If $\mathscr{F}$ is a family of subtrees of T that pairwise intersect, then $\bigcap_{F \in \mathscr{F}} V)(F) \neq \varnothing$.

More generally, we have the following:

Theorem 2.19: Helly Property for Trees

Let $\mathscr{F}$ be a collection of subtrees of a tree T. For any integer $k \ge 1$, either

1. there exist k vertex-disjoint trees in $\mathscr{F}$, or
2. there exists a set X of $< k$ vertices in T that intersects each tree in $\mathscr{F}$.

When does a graph contain k vertex-disjoint cycles?

Not if it has cycles, but we can delete $k-1$ vertices to get a forest. Is this in essence the only reason?

Theorem 2.20: Erdős-Pósa, 1965

For every integer $k \geq 1$, there exists $f(k)$ such that every graph G contains

- k vertex-disjoint cycles, or
- $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G - X$ is a forest.

Indeed, Erdős and Pósa showed that $f(k) = \Theta(k \log k)$.

Proposition 2.21

If H is a graph of minimum degree 3 and C is a shortest cycle of G, then $|C| \leq 2\log|V(G)|$.

Let $r_k := \log k + \log\log k + 4$ and let $s_k := \begin{cases} 4kr_k & k \geq 2 \\ 1 & k = 1 \end{cases}$

Lemma 2.22

If H is a cubic multigraph with $|V(H)| \geq s_k$, then H contains k vertex-disjoint cycles.

Proof:
By induction on k. If $k = 1$, trivial. Assume $k \geq 2$.

Let C be a shortest cycle of G. By Proposition 2.21, (or if H is not a graph and hence $|C| = 2$), then $|C| \leq 2\log|V(H)|$.

Claim $H - C$ contains a subdivision of a cubic multigraph H' with $|V(H')| \geq |V(H)| - 2|C|$.

Calculations show that $|V(H')| \geq |V(H)| - 2|C| \geq s_{k-1}$. By induction H' has $k-1$ vertex-disjoint cycles $C_1, \ldots, C_{k-1}$. Then $C, C_1, \ldots, C_{k-1}$ are k vertex-disjoint cycles of H as desired.

Now let's prove the claim.

Let $m := |E(C, H - C)|$. Since H is cubic, then $m \leq |C|$. Consider bipartitions (V_1, V_2) of H beginnign with $V_1 := V(C)$. While $H[V_2]$ has a vertex of degree at most 1: move it to V_1 (obtainin a biparititon with strictly fewer crossing edges).

Suppose n moves are done, then $|E(V_1, V_2)| \leq m - n$. Hence $H[V_2]$ has min degree ≥ 2 and $\leq m - n$ vertices of degree 2. Let H' be obtained from $H[V_2]$ by surpressing vertices of degree 2. Then

$$|V(H')| \geq |V(H)| - |C| - n - (m - n) \geq |H| - 2|C|$$

as desired. □

Proof of Erdős-Pósa:
Let $f_k := s_k + k - 1$. We may asume G has a cycle (as otherwise (2) holds with $X = \varnothing$). Let H be a maximal subgraph of G in which every vertex has degree 2 or 3, and U be the set of degree 3 vertices of H. By Lemma, we may assume $|U| < s_k$ (otherwise (1) holds).

Let $\mathscr{C}$ be the set of all cycles that avoid U and meet H in exactly one vertex, and Z be the set of these vertices. For all $z \in Z$, pick a cycle $C_z \in \mathscr{C}$ that contains z, and $\mathscr{C}' := \{C_z : z \in Z\}$.

Let $\mathscr{D}$ be the set of 2-regular components of H that avoid Z. Pick $x_D \in V(D)$ for all $D \in \mathscr{D}$. Let $X' := Z \cup \{x_D : D \in \mathscr{D}\}$.

Note that $\mathscr{C}' \cup \mathscr{D}$ is a set of vertex-disjoint cycles. If $|\mathscr{C}' \cup \mathscr{D}| \geq k$, then (1) holds. So we may assume

$|X'| \leq |\mathscr{C}' \cup \mathscr{D}| \leq k-1$.

Let $X = X' \cup U$. It suffices to show $G - X$ is a forest. Suppose not. Let C' be a cycle in $G - X$. But then $C' \cap V(H) \neq \varnothing$, otherwise $H \cup C'$ is more maximal, and $|V(C') \cap V(H)| \geq 2$ as $C' \notin \mathscr{C}$. So there exists an H-path between two vertices of $H - X$ and hence $H \cup P$ is more maximal. □

Erdős-Pósa property

A connected graph H has **Erdős-Pósa property** if for all $k \geq 1$, there exists $f(k)$ such that every graph G contains

- k vertex-disjoint H-models, or
- $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G - X$ is H-minor-free.

Since K_3 models are precisely cycles, we have:

Erdős-Pósa Theorem (Equivalent Formulation)

K_3 has the Erdős-Pósa property.

Which graphs have the Erdős-Pósa property?

We have the following Corollary of the Grid Theorem:

Corollary 2.23: Robertson and Seymour

H has the Erdős-Pósa property if and only if H is planar.

Proof:
Suppose H is not planar. Let Σ be a surfaceof minimum genus in which H embeds. Then the arbitrarily fat hexagonal grids G of Σ contain an H model; no two disjoint H models; no set $X \subseteq V(G)$, $|X| \leq f(k)$ where $G - X$ has no H model.

Suppose H is planar. Let H_k be the disjoint union of k-copies of H. If G contains H_k s a minor, then (1) of Erdős-Pósa holds. So we may assume G is H_k-minor-free.

Since H_k is planar, by the Grid Theorem, $\text{tw}(G) \leq w_{H_k}$ for some constant w_{H_k}. Hence there exists a tree-decomposition $(T, \mathscr{V})$ of G of width $\leq w_{H_k}$.

For every H-model F in G, let $T_F := \{t \in T : V(F) \cap V_t \neq \varnothing\}$. Since H is connected, T_F is a subtree of T. Let $\mathscr{F} := \{T_F : F \text{ is an } H \text{ model}\}$. If there exist k vertex-disjoint trees in $\mathscr{F}$, the (i) of Erdős-Pósa holds. Otherwise by General Helly Property Lemma, there exists $X' \subseteq V(T)$ with $|X'| < k$ intersecting all subtrees in $\mathscr{F}$. Then $X := \bigcup_{t \in X'} V_t$ where $|X| \leq (k-1) \cdot (w_{H_k} + 1)$ satisfies (ii) of Erdős-Pósa. □

2.6 The Graph Minor Structure Theorem

path-decomposition

A **path-decomposition** of a graph G is a tree-decomposition $(T, \mathscr{V})$ where T is a path.

path-width

The **path-width** of a graph G, denoted $\mathrm{pw}(G)$, is the minimum width of a path-decomposition of G.

Theorem (Robertson and Seymour)

A graph has large path-width if and only if it contains a large complete binary tree as a minor.

Corollary (Roberston and Seymour)

H-minor-free graphs have bounded path-width if and only if H is a forest.

Now recall the game of cops and robbers. A game with k cops and an invisible robber. Equivalently cleaning a graph of a "plague" which infects along edges. A monotonic winning cop strategy is equivalent to a path-decomposition of width $k-1$.

Theorem (LaPaugh 1982, Kirousis and Papadimitriou 1986)

k cops have a monotonic winning strategy if and only if they have a winning strategy.

adhesion

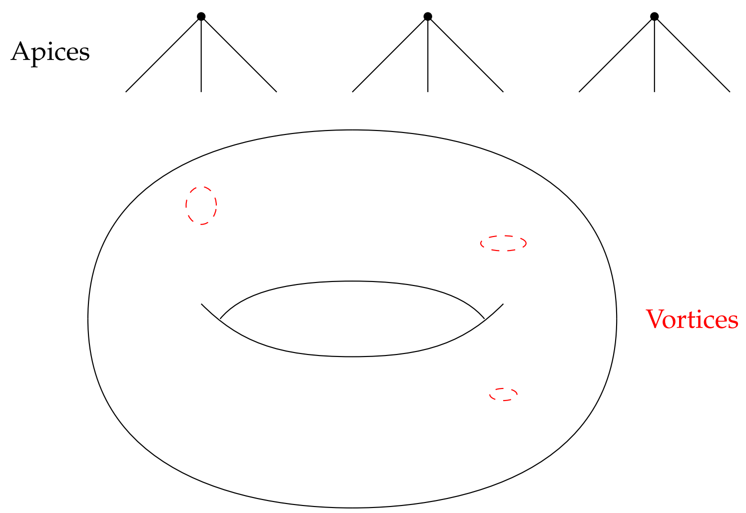
Let G be a graph and let $(T, \mathcal{V})$ be a tree-decomposition of G. For $t_1 t_2 \in E(T)$, the **adhesion set** for t_1 and t_2 is $V_{t_1} \cap V_{t_2}$. The **adhesion** of the decomposition is the maximum size of its adhesion sets.

torso

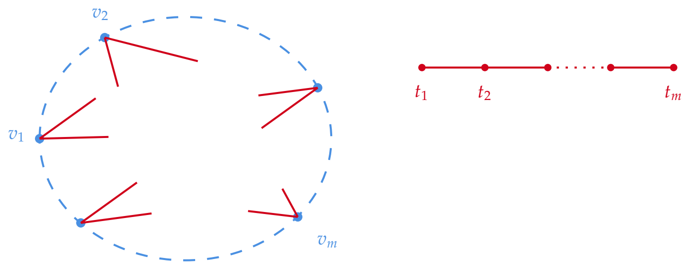
For $t \in V(T)$, the **torso** of the decomposition at t is the graph obtained from $G[V_t]$ by addding complete subgraphs to its adhesion sets.

k-nearly embeddable

A graph H is **k-nearly embeddable** in a surface Σ if it can be embedded in Σ except for $\leq k$ apices (i.e., allowed to delete k vertices), and $\leq k$ vortices.



A **vortex** is a disc that is cut out from Σ such that some of the vertices $v_1, \ldots, v_m$ of G are on the boundary of the disc and attached is a graph H with a path-decomposition $(T, \mathcal{V})$ of width at most k where $T = t_1 \ldots t_m$ is a path, and $v_i \in V_{t_i}$ for all $i \in [m]$.



Theorem 2.24: The Graph Minor Structure Theorem (Roberston and Seymour 2003)

For all $t \geq 5$, there exists k such that every graph with no K_t minor has a tree-decomposition whose torsos are k-nearly embeddable in a surface in which K_t does not embed.

Does the converse hold? Not necessarily, but qualitatively. Every graph that has a tree-decomposition whose torsos are k-nearly embeddable in a surface in which K_t does not embed, does not contain $K_{f(t)}$ as a minor for some function $f(t)$.

Fix k. let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be vertices in a graph G. When do there exist k internally disjoint paths from s_i's to the t_i's?

The 2 Paths Theorem (Seymour 1980)

There does not exist two disjoint paths from s_1 to t_1 and s_2 to t_2 if and only if G can be obtained from a planar graph H with s_1, s_2, t_1, t_2 on the outer face of H in that order via 3-sums with other graphs.

Theorem (Robertson and Seymour)

For every fixed k, deciding the disjoint paths problems can be done in polynomial time.

Proof Idea:
Case 1: G has bounded tree-width. Use dynamic programming.

Case 2: G has a large grid minor. Delete an 'irrelevant' edge in the center of the grid. □

Corollary 2.25

For every fixed graph H, deciding if a graph G contains H as a minor can be done in polynomial time.

Recall the following consequence of the Grid Theorem:

Theorem

If H is planar, then the class of H-minor-free graphs has bounded tree-width.

Corollary

The class of H-minor-free graphs is well-quasi-ordered.

Theorem 2.26

The set of forbidden minors for the class of graphs embeddable in a surface Σ is finite.

Proof sketch:

Let $H_0, H_1, \ldots$ be the forbidden minors.

Case 1: Infinitely many H_i do not contain a large grid as a minor. Then they have bounded tree-width and hence are WQO, a contradiction.

Case 2: WLOG H_0 contains a large grid as minor. Let e be an edge in the center of the grid. Now $H_0 - e$ embeds in Σ, but then so does H_0 (by re-embedding the grid part). $\square$

Now go back to the Graph Minor Theorem.

Graph Minor Theorem

Graphs are well-quasi-ordered by the graph minor relation.

Proof Sketch:

Let $G_0, G_1, \ldots$ be a bad sequence.

Case 1: G_0 is planar. Then G_0-minor-free graphs have bounded tree-width and hence are WQO.

Case 2: G_0 is not planar. By the Graph Minor Structure Theorem, G_0-minor-free graphs have a tree-decomposition whose torsos are k-nearly embeddable in a surface in which K_t does not embed.

There exists an infinite subsequence $H_0, H_1, \ldots$ that all embed in the same surface Σ. By induction on genus of Σ. If $\Sigma = S_0$, then H_0 is planar, done. So $\Sigma \neq S_0$. Excluding H_0 bounds the edge-width of $H_1, H_2, \ldots$. Hence they all have bounded non-contractible cycles. Cut along these to get new graphs.

Case A: cycles are non-separating. Then smaller genus graphs plus bounded apices.

Case B: cycles are separating. Then pairs of smaller genus graphs. $\square$

2.7 Connectivity and Minors

Does large average degree force a large complete minor? Kostochka 1982, Thomason 1984 showed that if G has average degree $\Omega(t\sqrt{\log t})$, then G contains a K_t minor. Does large average degree force a large complete subdivision? Bollobás and Thomason (1998) showed that If G has average degree $\Omega(t^2)$, then G contains a subdivision of K_t.

Does large average degree force a subgraph of large connectivity? Let the **density** of a graph G, denoted $d(G) = \frac{|E(G)|}{|V(G)|}$.

Theorem 2.27: Mader 1972

If G is a graph $d(G) \geq 2k$, G contains a $(k+1)$-connected subgraph H with $d(H) > d(G) - k$.

Proof:

Let $\gamma := d(G)$ and let H be a subgraph of G such that $|V(H)| \geq 2k$, and $|E(H)| > \gamma \cdot (|V(H)| - k)$, and subject to that $|V(H)|$ is minimized. Note that H exists since G is a such a graph.

$|V(H)| > 2k$ since otherwise $|E(H)| \leq \binom{2k}{2} < \gamma \cdot (2k - k)$.

Case 1: $\delta(H) \leq \gamma$. There exists $v \in V(H)$ with $d_H(v) \leq \gamma$. Let $H' := H - v$. Then $|V(H')| \geq 2k$ and $|E(H')| > \gamma \cdot (|V(H)| - k) - \gamma$, contradicting minimality of H.

Case 2: $\delta(H) > \gamma$. Hence $|V(H)| \geq \gamma$, and $|E(H)| > \gamma \cdot |V(H)| - \gamma k \geq \gamma \cdot |V(H)| - |V(H)| \cdot k$, and $d(H) > \gamma - k$.

It remains to show H is $(k+1)$-connected.

Suppose not, then let U_1, U_2 be a separation of order $\leq k$. Let $H_i := H[U_i]$ for all $i = 1, 2$. Note that $|V(H_1)|, |V(H_2)| \geq \gamma \geq 2k$. By minimality of H, $|E(H_i)| \leq \gamma \cdot (|V(H_i)| - k)$. Hence

$$\begin{aligned} |E(H)| &\leq |E(H_1)| + |E(H_2)| \\ &\leq \gamma \cdot (|V(H_1)| + |V(H_2)| - 2k) \\ &\leq \gamma \cdot (|V(H)| - k) \end{aligned}$$

a contradiction (we used $|V(H_1) \cap V(H_2)| \leq k$). $\square$

Does a graph of huge chromatic number contain a subgraph of large chromatic number and large connectivity?

Alon, Kleitman, Thomassen, Saks, Seymour (1987): $\chi \geq \Omega(k^3) \implies k$-conencted subgraph with $\chi \geq k$. Girão and Narayanan (2020+): For every positive integer k, if G is a graph with $\chi(G) \geq 7k$, then G contains a k-connected subgraph H with $\chi(H) \geq \chi(G) - 6k$.

Fix k. Let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be vertices in a graph G. When do ther exist k internally disjoint paths from the s_i's to the t_i's?

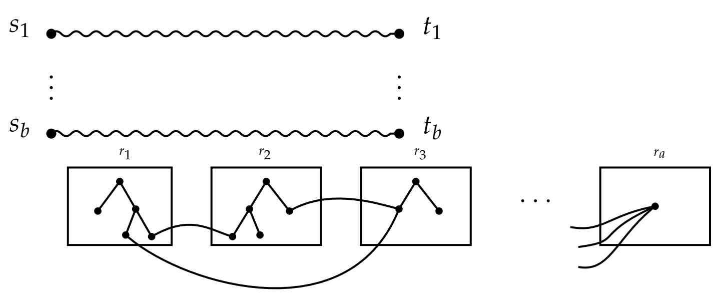
linked

A graph G is k-**linked** if for any set of vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ of G, there exist internally vertex-disjoint paths $(P_i : i \in [k])$ from s_i to t_i.

Clearly k-linked implies k-connected. Does some converse hold? Bollobás and Thomason (1996) showed that if a graph G is $\Omega(k)$-connected, then G is k-linked. Best bound: $10k$ by Thomas and Wollan (2005).

woven

A graph G is (a, b)**-woven** if for every three sets of vertices $R = \{r_1, \ldots, r_a\}$, $S = \{s_1, \ldots, s_b\}$, $T = \{t_1, \ldots, t_b\}$ in $V(G)$, there exists a K_a model in G rooted at R internally vertex-disjoint from a set of internally vertex-disjoint paths $(P_i : i \in [k])$ from s_i to t_i.



Norin and Song (2019+) showed that if a graph G is $\Omega(a\sqrt{\log a} + b)$-connected, then G is (a, b)-woven.

When does a graph of density d have a minor of density D?

Obstruction: small graphs $|V(G)| < \frac{D^2}{d}$ and disjoint unions of such graphs!

Theorem: Density Increment

For all $s \geq 1$, there exists $g(s)$ such that if G is a graph with $d(G) > 0$, and we let $D = s \cdot d(G)$, then G contains at least one of the following:

1. a minor J with $d(J) \geq D$, or
2. a subgraph H with $|V(H)| \leq g(s) \cdot \frac{D^2}{d(G)}$ and $d(H) \geq \frac{d(G)}{g(s)}$.

Latest result: Postle (2020+), $g(s) = O\big((1 + \log s)^6\big)$.

Theorem (Kuhn and Osthus 2003)

If G has girth at least five and average degree at least $\tilde{\Omega}(t^{2/3})$, then G contains a K_t minor.

Proof:
Let $D = \Omega(t\sqrt{\log t})$. By Density Increment Theorem, either

1. G contains a minor of density D and hence a K_t minor by Kostochka-Thomason, or
2. there exists a subgraph H of G with $|V(H)| \leq \tilde{O}(t^{4/3})$ and minimum degree at least $\tilde{\Omega}(t^{2/3})$, a contradiction since H has girth at least 5. □

Kuhn and Osthus (2005) showed that if G has no $K_{s,s}$ subgraph and average degree at least $\tilde{\Omega}(t^{2(s-1)/(2s-1)})$, then G contains a K_t minor.

2.8 Hadwiger's Conjecture

Hadwiger's Conjecture (1943)

For all $t \geq 1$, every graph with no K_t minor is $(t-1)$-colorable.

Hadwiger (1943) and independently Dirac (1952) proved it for $t \leq 4$. Wagner (1937) showed that the $t = 5$ case is equivalent to the Four Color Theorem, which was proved by Appel and Haken in 1977. Robertson, Seymour and Thomas (1993) showed that the $t = 6$ case is also equivalent to 4CT, and hence true. Open for $t \geq 7$.

Theorem (Kostochka 1982, Thomason 1984)

Every graph with no K_t minor is $O(t\sqrt{\log t})$-degenerate and hence $O(t\sqrt{\log t})$-colorable.

Until recently, the only improvements (Thomason 2001, Wood 2013, Kelly and Postle 2019) had been in the constant factor.

Theorem (Duchet and Meyniel 1982)

For all $t \geq 2$, every graph G with no K_t minor has an independent set of size at least $\frac{|V(G)|}{2(t-1)}$.

Proof:
By induction on t. $t = 2$ is trivial. We may assume G is connected.

Let A, B be disjoint subsets of $V(G)$ such that A is an independent set, $|A| \geq |B|$, and $G[A \cup B]$ is connected, and subject to that $|A|$ is maximized. We may assume $|A| \leq \frac{|V(G)|}{2(t-1)}$ otherwise we are done.

Case 1: for all vertex of $G' := G - (A \cup B)$ has a neighbor in $A \cup B$. Then G' has no K_{t-1} minor. By induction

$$\alpha\left(G'\right) \geq \frac{|V\left(G'\right)|}{2(t-2)} \geq \left(1 - \frac{1}{(t-1)}\right) \frac{|V(G)|}{2(t-2)} = \frac{|V(G)|}{2(t-1)}$$

Case 2: Otherwise. There exists $u \in V(G')$, $d(u, A \cup B) = 2$. Let $v \in N(u) \cap N(A \cup B)$. Then $A' := A \cup \{u\}, B' := B \cup \{v\}$ contradict maximality of A, B. □

Reed and Seymour (1998) showed that for all $t \geq 2$, every graph G with no K_t minor satisfies $\chi_f(G) \leq 2(t-1)$.

A d-**defective coloring** of a graph G is an improper coloring of G where each color class has maximum degree at most d.

Edwards, Kang, Kim, Oum, Seymour (2015) showed that for all $t > 0$, there exists d such that if G has no K_t minor, then G has a d-defective coloring with $t-1$ colors. Defect is $O(t^2 \log t)$. Improved to defect $t-2$ by van den Heuvel and Wood (2018).

A c-**clustered coloring** of a graph G is an improper coloring of G where each color class has maximum component size at most c.

Theorem

For all $t \geq 0$, there exists c such that if G has no K_t minor, then G has a c-clustered coloring with $f(t)$ colors.

Latest result: $f(t) = t - 1$ (announced by Dvořák and Norin)

To color small graphs, we have corollary (of Duchet-Meyniel): If G is a graph with no K_t minor with $|V(G)| \leq t \cdot poly(\log t)$, then $\chi(G) \leq O(t \log \log t)$.

Let $f(t) := O((\log \log t)^6)$. Then we have Corollary (of Density Increment Theorem and Mader's): for all $k \geq t$, if G is a graph with $d(G) \geq k \cdot f(t)$ and G contains no K_t minor, then G contains a k-connected subgraph H with $|V(H)| \leq t \cdot f(t) \cdot \log t$.

Another corollary: If G is a graph with no K_t minor and $\chi(G) \geq k \cdot f(t) + 2t \log f(t) + 6t \log r$, then G contains r vertex-disjoint k-connected subgraphs $H_1, \ldots, H_r$ with $|V(H_i)| \leq t \cdot f(t) \cdot \log t$ for every $i \in [r]$.

Then using previous results, we can build a minor when $\chi = \Omega(t(\log t)^{1/4} \cdot f(t))$.

chromatic separable

Let $s \geq 0$. A graph G is s-**chromatic-separable** if there exist two vertex-disjoint subgraphs H_1, H_2 of G such that for all $i = 1, 2$, $\chi(H_i) \geq \chi(G) - s$, and that G is s-**chromatic-inseparable** otherwise.

Two cases: always separable vs. inseparable.

Lemma (Always Separable Case)

Let $s \geq t$. If G is a graph with $\chi(G) \geq \Omega(s \log \log t)$ and every subgraph H of G with $\chi(H) \geq \frac{\chi(G)}{2}$ is s-chromatic-separable, then G contains a K_t minor.

Lemma (Inseparable Case)

Let $s = \Omega(t \log\log t)$. If G is a s-chromatic-inseparable graph with $\chi(G) \geq (t \cdot (f(t) + \log\log t))$, then G contains a K_t minor.

Always separable case: recursively build the minor. Build three $K_{2t/3}$ models and link them.

Inseparable case. Sequentially build the minor. Successively build up a K_t model by adding $\sqrt{\log t}$ new vertices (one new column) at a time.

Proof of Main Theorem:

Let $s = \Omega(t \log\log t)$. As $\chi(G) \geq \Omega(t(\log\log t)^2) \geq \Omega(s \log\log t)$, we have by the Always Separable Case Lemma that there exists a subgraph H of G with $\chi(H) \geq \frac{\chi(G)}{2}$ is s-chromatic-inseparable.

Since $\chi(H) \geq \Omega(t(\log\log t)^6) \geq \Omega(t \cdot (f(t) + \log\log t))$, we have by the Inseparable Case Lemma that H contains a K_t minor. $\square$

odd minor

A graph G contains H as an **odd minor** if a graph isomorphic to H can be obtained from a subgraph G' by contracting a set of edges forming a cut in G'.

Odd Hadwiger's Conjecture (Gerards and Seymour)

For all $t \geq 1$, every graph with no odd K_t minor is $(t-1)$-colorable.

Geelen, Gerards, Reed, Seymour and Vetta (2008): $O(t\sqrt{\log t})$-colorable. Norin and Song (2019+) - $O(t(\log t)^\beta)$-colorable for every $\beta > \frac{1}{4}$. Postle (2020+): $O(t(\log\log t)^6)$-colorable.

Also we have some generalizations of Hadwiger's Conjecture for list coloring. Voigt (1993) showed there exists a planar graph that is not 4-list-colorable (Hence List Hadwiger's is false). Postle (2020+) showed $O(t(\log\log t)^6)$-list-colorable.

3 Extremal Graph Theory

The field of extremal graph theory concerns the following:

- How does the value of a global parameter (e.g. edges, chromatic number) influence the existence of a local substructure (e.g. complete subgraph, complete minor)?
- (Contrapositive) What is the maximum value of a global parameter (e.g. edges, chromatic number) when excluding a given substructure (e.g. complete subgraph, complete minor)?

Generally there are two subareas of extremal:

- Sparse - where $|E(G)| = \Theta(|V(G)|)$ - e.g., Hadwiger's conjecture, Kostochka-Thomason, etc.
- Dense - where $|E(G)| = \Theta(|V(G)|^2)$ - e.g., Turán's Theorem, Regularity Lemma, etc.

What is the maximum number of edges in a triangle-free graph on n vertices.

Theorem (Mantel 1907)

The maximum number of edges in a triangle-free graph on n vertices is

$$\left\lceil \frac{n}{2} \right\rceil \cdot \left\lfloor \frac{n}{2} \right\rfloor$$

The graphs attaining such a bound for an extremal problem are called the **extremal examples** for that problem. What are the extremal examples for this problem? Certainly $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ is one such graph; and one can prove that is it the only extremal example.

3.1 Turán's Theorem

What is the maximum number of edges in a K_r-free graph on n vertices?

Theorem (Turán 1941)

The number of edges in a K_r-free graph on n vertices is at most

$$\left(1 - \frac{1}{r-1}\right) \binom{n}{2}$$

Is this tight? What are the extremal examples?

Turán number

The **Turán number** of a graph H, denoted $\text{ex}(n, H)$, is the maximum number of edges in a graph on n vertices that does not contain H as a subgraph. An H-free graph on n vertices with $\text{ex}(n, H)$ edges is called extremal for n and H.

Complete $(r-1)$-partite graphs would be nice candidates to be extremal for K_r-free. Which such graphs have the most edges?

Turán graph

The unique complete $(r-1)$-partite graphs on $n \geq r-1$ vertices whose partition sets differ by at most 1 are called **Turán graphs**. We denote the graph by $T^{r-1}(n)$ and its number of edges by $t_{r-1}(n)$.

Theorem 3.1: Turán's Theorem (1941)

For all integers r, n with $r > 1$, every K_r-free graph G with n vertices and $\text{ex}(n, K_r)$ edges is a $T^{r-1}(n)$.

What is $t_{r-1}(n)$? Let $n = (r-1)m + k$ where $k \in \{0, \ldots, r-2\}$. $T^{r-1}(n)$ has k parts of size $m+1$ and $r-1-k$ parts of size m. Hence

$$\begin{aligned}
2|E(G)| &= k(m+1)(n-(m+1)) + (r-1-k)m(n-m) \\
&= (k(m+1) + (r-1-k)m)(n-m) - k(m+1) \\
&= n(n-m) + k(m+1) \\
&\leq n^2\left(1 - \frac{1}{r-1}\right)
\end{aligned}$$

since $n(k+m) + k \geq n/(r-1)$.

Proof (first proof of Turán's Theorem):

By induction. If $n \leq k-1$, then $T^{r-1}(n) = K_n$ as desired. So $n \geq r$. Since G is an edge-maximal K_r-free graph, G contains a K_{r-1} subgraph K. G_K has at most $t_{r-1}(n-r+1)$ edges by induction. Each vertex has at most $r-2$ neighbors in K since G is K_r-free. Hence

$$|E(G)| \leq t_{r-1}(n-r+1) + (n-r+1)(r-2) + \binom{r-1}{2} = t_{r-1}(n)$$

The equality follows by removing one vertex from each part of the Turán graph.

Since G is extremal for K_r, equaility holds. Hence every vertex in $G-K$ has exactly $r-2$ neighbors in K. Let $V(K) = \{x_1, \ldots, x_{r-1}\}$. For all $i \in [r-1]$, let $V_i := \{v \in V(G) : vx_i \notin E(G)\}$. Each V_i is independent since G is K_r-free. G is $(r-1)$-partite since the V_i are independent and partition $V(G)$. Since $T^{r-1}(n)$ is the unique $(r-1)$-partite graph with n vertices and maximum number of edges, we have $G = T^{r-1}(n)$ as desired. $\square$

vertex duplication

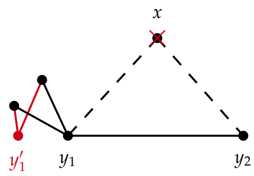
Let G be a graph. **Duplicating** a vertex $v \in V(G)$ means adding a new vertex v' adjacent to exactly the neighbors of v but not v itself.

Proof (second proof of Turán's Theorem):

The Turán graph $T^k(n)$ has (uniquely) the most edges among complete k-partite graphs on n vertices. $T^{r-1}(n)$ has more edges than $T^k(n)$ for all $k < r-1$. So if suffices to show G is complete multipartite.

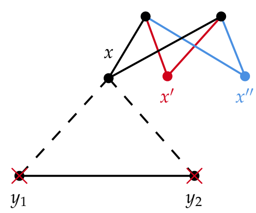
Suppose not. This means that in G non-adjacency is not an equivalence relation. that is, there exist $y_1, x, y_2 \in V(G)$ such that $y_1x, y_2x \notin E(G)$ and $y_1y_2 \in E(G)$.

Case 1: $d(y_1) < d(x)$ or $d(y_2) > d(x)$.



WLOG $d(y_1) > d(x)$. Let G' be obtained from $G - x$ by duplicating y_1 with a new vertex y_1'. $|E(G')| > |E(G)|$ since $d_{G'}(y_1') = d_G(y_1) > d_G(x)$. G' is K_r-free since a K_r, H in G' must use y_1', but then $H - y_1' + y_1$ is a K_r in G. Hence G' contradicts the maximality of G.

Case 2: $d(y_1) \leq d(x)$ and $d(y_2) \leq d(x)$.



Let G' be obtained from $G - \{y_1, y_2\}$ by duplicating x twice with two new vertices x', x''. Since we delete $d_G(y_1) + d_G(y_2) - 1$ edges and add $2 \cdot d_G(x)$ edges, then $|E(G')| > |E(G)|$. Note that G' is K_r-free (since a K_r, H in G' must use exactly one of x' or x'', but then $H - x' + x$ or $H - x'' + x$ is a K_r in G). Hence G' contradicts the maximality of G.

□

3.2 Ramsey's Theorem

Can we find order in chaos? Structure where there is none?

What substructures are necessarily present in large enough graphs? Does every large enough graph contain a large clique or independent set? This question and related ones form a branch of graph theory called **Ramsey Theory**.

Theorem (Ramsey 1930)

For all positive integers r, there exists a number n such that every graph G with at least n vertices contains either a K_r or $\overline{K_r}$ as an induced subgraph.

Recall $\bar{G}$ denote the complement: the complement or inverse of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in G.

Ramsey number

The **Ramsey number** of a graph H, denoted $R(H)$, is the minimum integer n such that every 2-coloring of the edges of K_n contains a monochromatic copy of H.

Ramsey's Theorem is thus equivalent to the statement that $R(K_r)$ exists for all r as we can simply color the edges red and the non-edges blue. Furthermore, it follows that $R(H)$ exists for all H (since if we find a monochromatic $K_{|V(H)|}$, we find a monochromatic H).

Proof of Ramsey's Theorem:

For $r = 1$, trivial. So we assume $r \geq 2$. Let $n = 2^{2r-3}$.

We claim that for all $i \in [2r-2]$, there exists $V_i \subseteq V(G)$ and $v_i \in V_i$ such that

(i) $|V_i| = 2^{2r-2-i}$, and

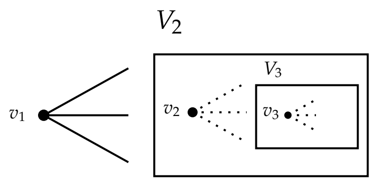
(ii) $V_i \subseteq V_{i-1} \setminus \{v_{i-1}\}$ for all $i \in \{2, \ldots, 2r-2\}$

(iii) v_{i-1} is adjacent to all vertices in V_i or to no vertex in V_i.

Proof:

By induction on i. For $i = 1$, let V_1 be a subset of $V(G)$ of size 2^{2r-3} and let $v_1 \in V_1$ arbitrarily.

So we assume $i \geq 2$. Then $|V_{i-1} \setminus \{v_{i-1}\}| = 2^{2r-1-i} - 1$. Hence v_{i-1} is either adjacent to 2^{2r-2-i} vertices in the set or non-adjacent to 2^{2r-2-i} vertices in the set. □



Among the $2r-3$ vertices $v_1, \ldots, v_{2r-3}$, at least $r-1$ vertices satisfy (iii) in the same way. That is, $\exists I \subseteq [2r-3]$ with $|I| = r-1$ such that for all $i \in I$ either v_i is not adjacent to V_{i+1}, or v_i is adjacent to all of V_{i+1}. But then $\{v_i : i \in I\} \cup \{v_{2r-2}\}$ induces a K_r or $\overline{K_r}$ in G as desired. □

Generalizations of Ramsey's Theorem

First generalization: c colors instead of two.

Second generalization: hypergraphs instead of graphs.

monochromatic

Let X be a set. We let $[X]^k$ denote the set of all k-subsets of X. Given a coloring of $[X]^k$, we say $Y \subseteq X$ is **monochromatic** if all elements of $[Y]^k$ have the same color.

Theorem 3.2

For all $k, c, r \geq 1$, there exists $n \geq k$ such that every n-set X and c-coloring of $[X]^k$ has a monochromatic r-subset.

We will derive this theorem from a third Generalization: the Infinite Version.

Theorem 3.3: Infinite Ramsey

Let k, c be positive integers and X an infinite set. Then every c-coloring of $[X]^k$ contains an infinite monochromatic subset.

Proof:

By induction on k with c fixed. For $k = 1$, trivial. So assume $k \geq 2$.

We claim that for all $i \in \{0, 1, \ldots\}$, $\exists X_i \subseteq X$ and $x_i \in X_i$ such that for all i:

(i) X_i is infinite

(ii) $X_{i+1} \subseteq X_i \setminus \{x_i\}$

(iii) all k-sets $\{x_i\} \cup Z$ with $Z \in [X_{i+1}]^{k-1}$ have the same color, which we associate with x_i.

Proof:
By induction on i. For $i = 0$, let $X_0 = X$ and pick $x_0 \in X_0$ arbitrarily. So assume $i \geq 1$. Construct a c-coloring of $[X_i \setminus \{x_i\}]^{k-1}$ by letting the color of Z be the same as $Z \cup \{x_i\}$. By induction on k, there exists an infinite monochromatic subset X_{i+1} as desired. □

Since c is finite, one of the c colors is associated with infinitely many x_i. These x_i form an infinite monochromatic subset of X as desired. □

Now we prove the finite version, Theorem 3.2: compactification.

Proof:
Suppose not. That is, the statement fails for some k, c, r. That is, for all $n \geq k$, there exists a c-coloring of $[n]^k$ with no monochromatic r-subset. Call such colorings bad. Let V_n be the set of all bad colorings of $[n]^k$. Note for $g \in V_n$, the restriction $f(g)$ of g to $[n-1]^k$ is also bad.

We claim that $\exists g_k, g_{k+1}, \ldots$ such that $g_n \in V_n$ and $f(g_n) = g_{n-1}$ for all $n > k$.

Proof:
We prove this with the stronger assumption that g_n extends to infinitely many elements of $\bigcup_{i \geq n} V_i$.

By induction on n. For $n = k$, trivial. So assume $n > k$. Since g_{n-1} extends to infinitely many elements, $\exists g_n \in V_n$ □

Define a c-coloring g of $[\{0, 1, \ldots\}]^k$ by letting $g(Y) := g_{\max Y}(Y)$. Now g is a bad-coloring contradicting Infinite Ramsey since every r-set S is contained in $[\max S]$. □

3.3 The Regularity Lemma

What substructures are necessarily present in large enough graphs? Regularity Lemma states that: Every large enough graph is approximately a random blowup of some fixed weighted graph.

density of the pair (X, Y)

Let G be a graph and X, Y disjoint subsets of $V(G)$. We denote by $\|X, Y\|$ the number of $X - Y$ edges. The **density** of the pair (X, Y) is

$$d(X, Y) := \frac{\|X, Y\|}{|X| \cdot |Y|}$$

epsilon-regular

Let G be a graph and A, B disjoint subsets of $V(G)$. We say the pair (A, B) is **ε-regular** if all $X \subseteq A, B \subseteq Y$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy

$$|d(X, Y) - d(A, B)| \leq \varepsilon$$

epsilon-regular partition

Let G be a graph. A partition $\{V_0, V_1, \ldots, V_k\}$ is an **ε-regular partition** if

(i) $|V_0| \leq \varepsilon |V(G)|$, and

(ii) $|V_1| = \cdots = |V_k|$, and

(iii) all but at most εk^2 of the pairs (V_i, V_j) with $i < j \in [k]$ are ε-regular.

We call V_0 the **exception set**.

Thus the partition is equitable (except for the exceptional set) and all but a small proportion of the pairs are ε-regular (though each pair may have a different density).

Regularity Lemma (Szemerédi 1976)

For every $\varepsilon > 0$ and integer $m \geq 1$, there exists an integer M such that every graph with at least m vertices admits an ε-regular partition $\{V_0, V_1, \ldots, V_k\}$ with $m \leq k \leq M$.

Note:
$V_1 = V(G)$ is an ε-regular partition for every ε. Yet $V_i = \{v_i\}$ where $V(G) = \{v_1, v_2, \ldots, v_{|V(G)|}\}$ is also ε-regular for every ε. Hence the beauty of the Regularity Lemma is that we can find an ε-regular partition with at least m parts, and at most M parts where M depends only on m and ε.

Let's first have an overview of the proof: we define a potential function on partitions of $V(G)$ that is a quadratoc measurement of the densities of the pairs. If at least εk^2 pairs are not ε-regular, then we **refine** the partition further increasing the density by a fixed constant. (e.g., $1^2 + 3^2 > 2^2 + 2^2$; $(d-\varepsilon)^2 + (d+\varepsilon)^2 \geq 2d^2 + 2\varepsilon^2$). The potential belongs to $[0,1]$ and hence this refinement happens only a fixed number of times. Thus the final number of parts depends only on the starting number of parts (m) and ε.

Quadratic Measurement

Let G be a graph. For disjoint sets A, B of $V(G)$ we define

$$q(A,B) := \frac{|A||B|}{|V(G)|^2} \cdot d(A,B)^2$$

For partitions $\mathscr{A}$ of A and $\mathscr{B}$ of B we set

$$q(\mathscr{A}, \mathscr{B}) := \sum_{A' \in \mathscr{A}, B' \in \mathscr{B}} q\left(A', B'\right)$$

For a partition $\mathscr{P} = \{V_1, \ldots, V_k\}$ of $V(G)$, we set

$$q(\mathscr{P}) := \sum_{i<j} q(V_i, V_j)$$

Now we deal with exceptional sets. Note that we only use exceptional sets to enforce that the partition is equitable.

Quadratic Measurement cont'd

For a partition $\mathscr{P} = \{V_0, V_1, \ldots, V_k\}$ of $V(G)$ with exceptional set V_0, we set

$$q(\mathscr{P}) := q(\tilde{\mathscr{P}}),$$

where

$$\tilde{\mathscr{P}} := \{V_1, \ldots, V_k\} \cup \{\{v\} : v \in V_0\},$$

that is the partition where we make each vertex in the exceptional set its own singleton part.

Proposition 3.4

$q(\mathscr{P}) \in [0, 1]$.

Proof:
Certainly $q(\mathscr{P}) \geq 0$ since q is always nonnegative. And

$$\begin{aligned} q(P) &= \sum_{i<j} q\left(V_i, V_j\right) \\ &= \sum_{i<j} \frac{|V_i|\,|V_j|}{|V(G)|^2} \cdot d\left(V_i, V_j\right)^2 \\ &\leq \frac{1}{|V(G)|^2} \cdot \sum_{i<j} |V_i|\,|V_j| \\ &= 1 \end{aligned}$$

□

Lemma 3.5

If $\mathscr{C}$ is a partition of C and $\mathscr{D}$ is a partition of D, then $q(\mathscr{C}, \mathscr{D}) \geq q(C, D)$.

Proof:
Cauchy-Schwarz. □

Corollary 3.6

If $\mathscr{P}'$ is a partition of $V(G)$ that refines a partition $\mathscr{P}$, then $q(\mathscr{P}') \geq q(\mathscr{P})$.

Proof:
Apply lemma to each refined pair. □

Lemma 3.7

If (C, D) is not an ε-regular pair, then there exist partitions $\mathscr{C} = \{C_1, C_2\}$ and $\mathscr{D} = \{D_1, D_2\}$ such that

$$q(\mathscr{C}, \mathscr{D}) \geq q(C, D) + \varepsilon^4 \cdot \frac{|C||D|}{|V(G)|^2}$$

Proof:
Let C_1, D_1 be a pair that witness irregularity. That is

$$|C_1| \geq \varepsilon|C|, |D_1| \geq \varepsilon|D| \text{ and } |d\left(C_1, D_1\right) - d(C, D)| > \varepsilon$$

Calculate (e.g., $(d+\varepsilon)^2\varepsilon^2 + \left(d - \frac{\varepsilon^3}{1-\varepsilon^2}\right)^2(1-\varepsilon^2) \geq d^2 + \varepsilon^4$). □

Lemma 3.8

If $\mathscr{P}$ is not an ε-regular partition and has k parts, then ther exists a partition $\mathscr{P}'$ on at most $k \cdot 4^k$ parts such that $q(\mathscr{P}') \geq q(\mathscr{P}) + \frac{\varepsilon^5}{2}$.

Proof:

There are at least εk^2 irregular pairs. Refine each irregular pair by previous lemma. Note that we do this simultaneously for all such pairs, so each part may be divided into 2^k smaller part (times 2 for each such pair it is in).

Calculate (e.g., $+\varepsilon^4 \cdot \frac{1}{k^2} \cdot \varepsilon k^2 = \varepsilon^5; 1/2$ comes from trash). Make equitable (sets of size $n/(k4^k)$). □

Now we are ready to prove the regularity lemma:

Proof:

Start with an arbitrary partition into m equitable parts plus remainder. Apply previous lemma at most $\frac{2}{\varepsilon^5}$ times. Since $q(\mathscr{P})$ cannot be greater than 1, this terminates with an ε-regular partition. The final number of parts depends only on m and ε. □

3.4 Erdős-Stone Theorem

Recall Turán's theorem. For every graph H, what is $\mathrm{ex}(n, H)$?

Let K_{s*r} denote the complete r-partite graph with s vertices in every part (equivalently this is $T^r(rs)$).

Theorem (Erdős and Stone 1946)

For all integers $r \geq 2$, $s \geq 1$ and every $\gamma > 0$, there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least $t_{r-1}(n) + \gamma \cdot n^2$ edges contains K_{s*r} as a subgraph.

Corollary (Erdős and Simonovits 1966)

For every graph H with at least one edges,

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}$$

Proof:

Let $r := \chi(H)$. But then H is not a subgraph of $T^{r-1}(n)$ for all n. Hence $\mathrm{ex}(n, H) \geq t_{r-1}(n)$. On the other hand, H is a subgraph of K_{s*r} where $s = |V(H)|$. Hence

$$\mathrm{ex}(n, H) \leq \mathrm{ex}(n, K_{s*r}) = t_{r-1}(n) + o(n^2)$$

□

Our proof of the Erdős-Stone theorem will not be the original but rather an easy application of a version of a corollary of the Regularity Lemma called the Blow-Up Lemma.

regularity graph

Let G be a graph with an ε-partition $\{V_0, V_1, \ldots, V_k\}$ where V_0 is exceptional and $|V_1| = \cdots = |V_k| = \ell$. The **regularity graph** R of G with parameters ε, ℓ and $d \in [0,1]$ is the graph where $V(R) := \{V_1, \ldots, V_k\}$, $E(R) := \{V_iV_j : (V_i, V_j)$ is an ε-regular pair of density $\geq d\}$.

If R is a regularity graph, we let R_s denote the s-**blowup of** R i.e., where each vertex V_i is replaced by a set of V_i^s of s vertices, and every edge is replaced by a complete bipartite graph.

Blow-Up Lemma

For all $d \in [0,1]$, $\Delta \geq 1$, there exists $\varepsilon_0 > 0$ such that if

- G is a graph, and
- R is a regularity graph of G with parameters ε, ℓ, d, and
- H is a subgraph of R_s with $\Delta(H) \leq \Delta$,

then H is a subgraph of G provided $\varepsilon \leq \varepsilon_0$ and $\ell \geq 2s/d^\Delta$.

Let G be a graph and $Y \subseteq V(G)$. We let

$$N_k(Y) := \{v \in V(G) : |N(v) \cap Y| \geq k\}$$

Lemma 3.9: ε-regular pair lemma

Let (A, B) be an ε-regular pair of density d. If $Y \subseteq B$ with $|Y| \geq \varepsilon|B|$, then

$$\left|A \setminus N_{(d-\varepsilon)|Y|}(Y)\right| < \varepsilon|A|$$

Proof:
Let $X := A \setminus N_{(d-\varepsilon)|Y|}(Y)$. Then $\|X, Y\| < |X| \cdot (d-\varepsilon)|Y|$. Hence $d(X,Y) < d - \varepsilon = d(A,B) - \varepsilon$. Since (A,B) is ε-regular and $|Y| \geq \varepsilon|B|$, we find that $|X| < \varepsilon|A|$. $\square$

Now we prove blow-up lemma.

Proof of Blow-Up Lemma:
Choose $\varepsilon_0 < d$ small enough so that

$$(d - \varepsilon_0)^\Delta - \Delta\varepsilon_0 \geq \frac{1}{2}d^\Delta$$

Let $V(H) = \{v_1, \ldots, v_{|V(H)|}\}$ and let $r(i)$ be such that $v_i \in V^s_{r(i)}$ in R_s.

We claim that for all $i \in \{0, \ldots, |V(H)\}$, there exist distinct $u_1, \ldots, u_i \in V(G)$ and $T^i_j \subseteq V_{r(j)}$ for all $j > i$ such that

(i) $H[\{v_1, \ldots, v_i\}] \subseteq G[\{u_1, \ldots, u_i\}]$, and

(ii) $T^i_j \subseteq \bigcap_{v_k \in N_H(v_j):k\in[i]} N(u_k)$, and

(iii) $|T^i_j| \geq (d-\varepsilon)^{|N_H(v_j)\cap\{v_1,\ldots,v_i\}|}|V_{r(j)}|$.

Blow-Up Lemma now follows from Claim with $i = |V(H)|$. Now we prove the claim.

By induction on i. For $i = 0$, $T_j^0 = V_{r(j)}$ works. So we may assume $i > 0$. Pick u_i in

$$S_i := T_i^{i-1} \bigcap_{v_j \in N_H(v_i): j > i} N_{(d-\varepsilon)|T_j^{i-1}|}(T_j^{i-1})$$

This only works since

$$|S_i| \geq \left((d-\varepsilon_0)^\Delta - \Delta\varepsilon_0\right)|V_{r(i)}| \geq \frac{1}{2}d^\Delta \ell \geq s,$$

and there are at most $s-1$ u_i in $V_{r(i)}$. Hence (i) holds.

Set $T_j^i := \begin{cases} T_j^{i-1} \cap N_G(u_i) & \text{if } v_i v_j \in E(H) \\ T_j^{i-1} & \text{otherwise} \end{cases}$

Hence (ii) holds.

Since $u_i \in N_{(d-\varepsilon)|T_j^{i-1}|}(T_j^{i-1})$ for all $j > i$ with $v_j \in N_H(v_i)$, we find that

- $|T_j^i| \geq (d-\varepsilon)|T_j^{i-1}|$ for all such j,
- $|T_j^i| = |T_j^{i-1}|$ for all other j.

Hence (iii) holds as desired. □

Now we prove the Erdős-Stone Theorem.

Proofof Erdős-Stone Theorem:

Let $d := \gamma$, $\Delta = \Delta(K_{s*r})$ and choose ε small enough and $\ell \geq 2s/d^\Delta$ large enough (possible since n is large). Let R be a regularity graph of G with k parts and parameter ε, ℓ and d.

Case 1: R contains K_r. Then by Blow-Up Lemma G contains $H := K_{s*r}$.

Case 2: R does not contain K_r. By Turán's Theorem, $|E(R)| \leq t_{r-1}(k) \leq (1 - \frac{1}{r-1})\binom{k}{2}$. But then $|E(G)|$ is at most (letting $n = |V(G)|$)

$$|E(R)|\ell^2 + \varepsilon n^2 + \varepsilon k^2 \ell^2 + d\binom{k}{2}\ell^2 \leq t_{r-1}(n) + \gamma n^2$$

a contradiction (where we added the trash, irregular pairs and low density pairs). □

An interesting consequence of the proof of the Blow-Up Lemma is: generally there is not just one copy of H in G but rather $\Omega(|V(G)|^{|V(H)|})$ copies of H in G.

Removal Lemma

For all $\varepsilon > 0$, there exists $\delta(\varepsilon, H) > 0$ such that if a graph G has at most $\delta|V(G)|^{|V(H)|}$ copies of H, then G can be made H-free by removing at most $\varepsilon|V(G)|^2$ edges.

Proof Sketch:

We may assume regularity graph R is H-free by proof of Blow-Up Lemma. Then make G H-free by removing the edges from trash, irregular pairs and low density pairs. □

4

Probabilistic Method

4.1 More on Ramsey Numbers

Ramsey number

The **Ramsey number** of a graph H, denoted $R(H)$, is the minimum integer n such that every 2-coloring of the edges of K_n contains a monochromatic copy of H.

Ramsey number

The **Ramsey number** of graphs G and H, denoted $R(G, H)$, is the minimum integer n such that every 2-coloring of the edges of K_n contains a red copy of G or a blue copy of H.

We let $R(k) := R(K_k)$ and $R(k, \ell) := R(K_k, K_\ell)$.

We have bounds for small numbers:

- $R(2) = 2$
- $R(3) = 6$
- $R(4) = 18$
- $42 \leq R(5) \leq 48$
- $102 \leq R(6) \leq 165$

4.1.1 Upper bounds

Theorem 4.1: Erdős and Szekeres 1960

$R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1)$

Proof:
Majority argument □

Corollary 4.2

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}$$

Corollary 4.3

$$R(k) \leq (1+o(1))\frac{4^{k-1}}{\sqrt{\pi k}}$$

Theorem (Conlon 2009)

$$R(k) \leq k^{-c\log k/\log\log k} \cdot 4^k$$

Theorem (Sah 2020+)

$$R(k) \leq k^{-c\log k} \cdot 4^k$$

Still no change in the base of the exponent from 4.

4.1.2 Lower bounds

Theorem (Erdős 1947)

$$R(k) \geq (1-o(1))\frac{\sqrt{2}k}{e}2^{k/2}$$

Still no change in the base of the exponent from $\sqrt{2}$.

$G(n,p)$

$G(n,p)$ is the random graph on n vertices where every edge is present independently with probability p.

Lemma 4.4

$$\Pr[G_{n,p} \text{ contains a } K_k] \leq \binom{n}{k}p^{\binom{k}{2}}$$

Proof (Erdős):

The probability that a fixed set U of k vertices is a K_k is exactly $\prod_{e\in\binom{U}{2}} p = p^{\binom{k}{2}}$ since the edges are present with probability p.

By the union bound, the probability that $G_{n,p}$ contains a K_k is at most the sum over the probabilities that a fixed U is a K_k.

Since there are $\binom{n}{k}$ choices for U, the lemma holds as desired. □

Similarly we have

Lemma 4.5

$\Pr[G_{n,p} \text{ contains an induced } \overline{K_k}] \le \binom{n}{k}(1-p)^{\binom{k}{2}}$

Proof:

Let $n = (1-o(1))\frac{k}{\sqrt{2}e}2^{k/2}$ and $p = 1/2$. Then

$$\begin{aligned}\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} &\approx \left(\frac{ne}{k}\right)^k 2^{-k(k-1)/2} \\ &= \left((1-o(1))2^{(k-1)/2}\right)^k 2^{-k(k-1)/2} \\ &= (1-o(1))^k < \frac{1}{2}\end{aligned}$$

By union bound,

$$\Pr[G_{n,p} \text{ contains a } K_k \text{ or an induced } \overline{K_k}] < \frac{1}{2} + \frac{1}{2} = 1$$

Thus there exists some graph on n vertices with no K_k or induced $\overline{K_k}$. □

Theorem (Chvátal, Rödl, Szemerédi, Totter 1983)

For all $\Delta \ge 1$, there exists $c > 0$ such that every graph with $\Delta(H) \le \Delta$ satisfies $R(H) \le c|V(H)|$.

Proof Sketch:

Apply Regularity Lemma with m large enough, ε small enough to obtain a regularity partition $(V_0, V_1, \ldots, V_k)$.

Let R be the regularity graph whose edges are the ε-regular pairs.

Since there are at most εk^2 irregular pairs and ε is small enough, then by Turán's Theorem, R contains a $K_{R(\Delta+1)}$, call it H.

Consider a 2-coloring of the edges of H: red if density $\ge 1/2$ and blue if density $< 1/2$.

By Ramsey's Theorem, there exists a monochromatic $K_{\Delta+1}$ subgraph of H, call it K.

Case 1: K is colored red. By Blow-Up Lemma, G contains a copy of H.

Case 2: K is colored blue. By Blow-Up Lemma, the complement of G contains a copy of H. □

4.2 Large Girth, Large Chromatic Number

Do there exist triangle-free graphs of arbitrarily large chromatic number?

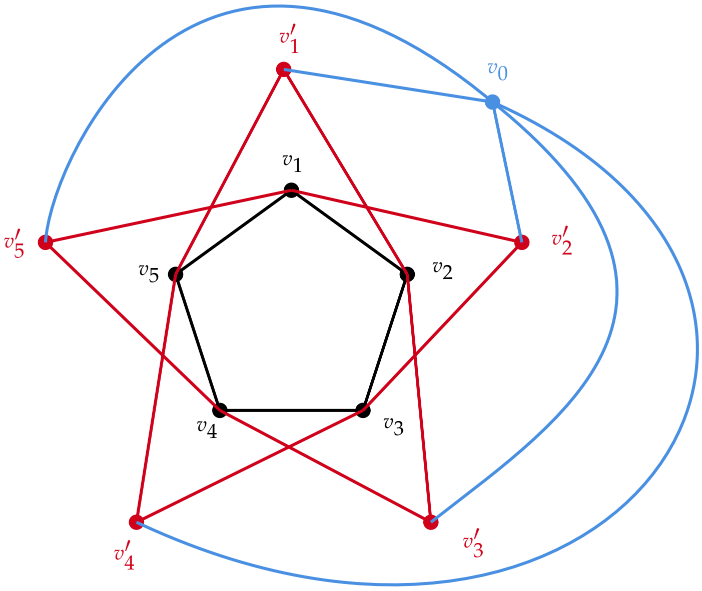
Yes. (Tutte 1947, Zykov 1949, Mycielski 1955)

Mycielski construction

Let G be a graph on n vertices and let $V(G) = \{v_1, \ldots, v_n\}$. The **Mycielski graph** of G is the graph G' with

$$\begin{aligned}V(G') &:= V(G) \cup \{v_1', \ldots, v_n'\} \cup \{v_0\} \\ E(G') &:= E(G) \cup \{v_i'v_j : v_iv_j \in E(G)\} \cup \{v_0v_i' : i \in [n]\}\end{aligned}$$

Let $G = C_5$. Then Mycielski graph G' of G is:



Note that if G is triangle-free, then so is G'. Moreover, $\chi(G') = \chi(G) + 1$.

Proof:
Let ϕ be a $\chi(G)$-coloring of G'. WLOG $\phi(v_0) = \chi(G)$.

So $\phi(v_i') \in [\chi(G) - 1]$ for all $i \in [n]$.

Let

$$\phi'(v_i) = \begin{cases} \phi(v_i') & \text{if } \phi(v_i) = \chi(G) \\ \phi(v_i) & \text{otherwise} \end{cases}$$

Then ϕ' is a $(\chi(G) - 1)$-coloring of G, a contradiction. □

Do there exist graphs of arbitrarily large girth and large chromatic number?

Yes by Erdős using the probabilistic method.

Theorem (Erdős 1959)

For all $r \geq 1$, there exists a graph of girth ≥ 4 and chromatic number $\geq r$.

Lovász (1968) gave the first explicit constructions.

4.2.1 Basic probability

Now it's time to review some basic probability theory...

Union bound:

$$\Pr[A_1 \cup A_2 \cup \cdots \cup A_k] \le \sum_{i=1}^{k} \Pr(A_i)$$

expectation

If X is a random real-valued valued variable with finite support, then the **expectation** of X is

$$\mathbb{E}[X] := \sum_i i \cdot \Pr[X = i]$$

Linearity of expectation: If $X = \sum_{i=1}^{r} X_i$, then

$$\mathbb{E}[X] = \sum_{i=1}^{r} \mathbb{E}[X_i]$$

Markov's Inequality: If $X \ge 0$, then for any $a > 0$

$$\Pr[X \ge a] \le \frac{\mathbb{E}[X]}{a}$$

Proof:
Since $X \ge 0$,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_i i \cdot \Pr[X = i] \\
&\ge \sum_{i \ge a} i \cdot \Pr[X = i] \\
&\ge a \cdot \sum_{i \ge a} \Pr[X = i] \\
&= a \cdot \Pr[X \ge a]
\end{aligned}$$

□

4.2.2 Random graphs

$G(n, p)$

$G(n, p)$ is the random graph on n vertices where every edge is present independently with probability p.

Lemma

$$\Pr[\omega(G_{n,p}) \ge k] \le \binom{n}{k} p^{\binom{k}{2}}$$

Lemma

$\Pr[\alpha(G_{n,p}) \geq k] \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$

Lemma

If $p \geq 4\frac{\ln n}{k}$ and $k \geq 2$, then
$$\Pr[\alpha(G_{n,p}) \geq k] \leq \frac{1}{n}$$

Proof:
Note that $(1-p) \leq e^{-p}$ for all p, and $e^{p(k-1)} \geq e^{pk/2} \geq n^2$. By lemma, we have
$$\begin{aligned}\Pr[\alpha(G_{n,p}) \geq k] &\leq \binom{n}{k}(1-p)^{\binom{k}{2}} \\ &\leq n^k e^{-p\binom{k}{2}} \\ &= \left(\frac{n}{e^{p(k-1)}}\right)^k \\ &\leq \frac{1}{n}\end{aligned}$$
□

Let $\#C_\ell(G)$ denote the number of cycles of length ℓ in a graph G. Let $(n)_\ell := n(n-1)\cdots(n-\ell+1)$.

Lemma

$\mathbb{E}[\#C_\ell(G_{n,p})] = \frac{(n)_\ell}{2\ell}p^\ell$

Proof:
Let $v_1, v_2, \ldots, v_\ell$ be a sequence in $V(G)$ of length ℓ. $\Pr[v_1 \ldots v_\ell$ is a cycle$] = p^\ell$. There are $(n)_\ell$ such sequences whie each cycle corresponds to exactly 2ℓ of these sequences. □

Corollary

If $p \leq n^{\frac{1}{2\ell}-1}$, then $\mathbb{E}[\#C_\ell(G_{n,p})] \leq \sqrt{n}$.

Proof:
By Lemma, $\mathbb{E}[\#C_\ell(G_{n,p})] = \frac{(n)_\ell}{2\ell}p^\ell \leq (np)^\ell \leq n^{1/2}$. □

Let $\#C_{\leq\ell}(G)$ denote the number of cycles of length $\leq \ell$ in G.

Corollary

If $p = n^{\frac{1}{2\ell}-1}$, then
$$\Pr[\#C_{\leq\ell}(G_{n,p}) \geq \frac{n}{2}] \leq \frac{2\ell}{\sqrt{n}}$$

Proof:
Markov's Inequality. □

There is no p that will give us that both:

- $\Pr[\chi(G_{n,p}) \leq r] < 1/2$ and
- $\Pr[G_{n,p}$ does not have girth at least $r] < 1/2$.

Instead we use a method called **alteration**: where in we deterministically alter a random outcome so as to obtain the desired object (graph).

Proof of Erdős' Theorem:

Let $\ell := r - 1$ and $p := n^{\frac{1}{2\ell}-1}$. By cycle lemma, $\Pr[\#C_{\leq \ell}(G_{n,p}) \geq \frac{n}{2}] \leq \frac{2\ell}{\sqrt{n}}$.

Let $k := \frac{n}{2r}$. Note $k \geq 2$ and $p \geq \frac{4 \ln n}{k}$ if n is large enough. By independence number lemma, $\Pr[\alpha(G_{n,p}) \geq k] \leq \frac{1}{n}$. Since n is large enough, $\frac{2\ell}{n} + \frac{1}{n} < 1$.

Hence there exists a graph G on n vertices with $\alpha(G) \leq \frac{n}{2r}$, and at most $n/2$ cycles of length $< r$. For each cycle C of length $< r$, choose $x_C \in V(C)$.

Let $X := \bigcup_C \{x_C\}$ and $G' := G - X$. Then G' has girth at least r. NOte $|V(G')| \geq n/2$ and thus

$$\alpha\left(G'\right) \leq \alpha(G) \leq \frac{n}{2r} \leq \frac{|V\left(G'\right)|}{r}$$

Hence $\chi(G') \geq r$ as desired. □

5

Flows

5.1 Flows

If G is a graph let $\vec{E}(G) := \{(e,x,y) : e = xy \in E(G)\}$ denote the set of directed edges of G.

H-circulation

If G is a graph and H is a finite Abelian group, then an H**-circulation** is a function $f : \vec{E}(G) \to H$ such that

- $f\big((e,x,y)\big) = -f\big((e,y,x)\big)$ for all $(e,x,y) \in \vec{E}(G)$, and
- $\sum\limits_{e=xy\sim x} f\big((e,x,y)\big) = 0$ for all $x \in V(G)$.

The second condition is called Kirchoff's law.

nowhere-zero

An H-circulation f is called **nowhere-zero** if $f\big((e,x,y)\big) \neq 0$ for all $(e,x,y) \in \vec{E}(G)$. We call such an H-circulation an H-flow.

If $X, Y \subseteq V(G)$, we let $\vec{E}(X,Y) := \{(e,x,y) : x \in X, y \in Y\}$ and $f(X,Y) = \sum\limits_{(e,x,y)\in\vec{E}(X,Y)} f\big((e,x,y)\big)$. Hence Kirchoff's law is equivalent to $f(x, V(G)) = 0$ for all $x \in V(G)$.

Proposition 5.1

If f is an H-circulation of G and $X \subseteq V(G)$, then $f(X,X) = 0$ and $f(X, V(G)) = 0$.

Proof:
$f(X,X) = 0$ since count edges both directions, $f(X, V(G)) = 0$ by Kirchoff's law. □

$P_G(H)$

If G is a graph and H is a group, we let $P_G(H)$ denote the number of H-flows of G.

Theorem 5.2: Tutte (1954), flow polynomial

For any graph G, there exists a polynomial $P_G(x)$ such that for any finite Abelian group H, $P_G(H) = P_G(|H|)$.

Corollary 5.3

For any graph G and group H, the number of H-flows of G depends only on $|H|$.

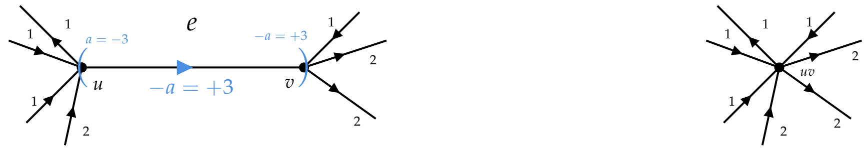
In particular, if an H-flow of G exists for any finite Abeliean group of order k, it exists for all of them.

Lemma 5.4: Contraction-Deletion

Let G be a graph. If $e \in E(G)$ is not a loop, then $P_G(H) = P_{G/e}(H) - P_{G-e}(H)$.

Proof:
Let $f(G, H)$ denote the set of H-flows of G. It suffices to prove that $f(G/e, H)$ is in bijection with $f(G, H) \cup f(G - e, H)$.

Let $e = uv$. Consider an H-flow of G/e.



Then there is a unique value for e such that the result is an H-circulation (i.e. satisfies Kirchoff's law).

Namely $f(e, u, v) := -\sum_{e'=uv':v'\neq v} f(e', u, v')$. If $f(e, u, v) = 0$, then f is an H-flow of $G - e$. If $f(e, u, v) \neq 0$, then f is an H-flow of G. This proves the lemma. □

Now we prove the flow polynomial theorem.

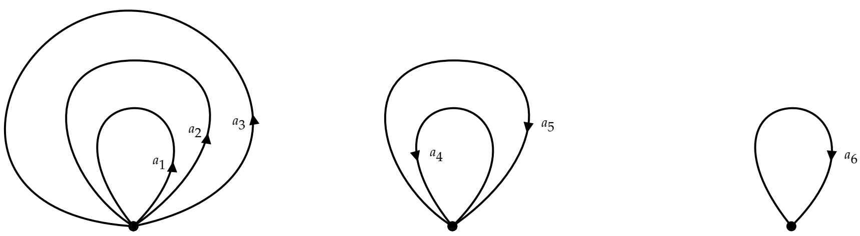
Proof:
By induction on $|E(G)|$.

If there exists $e \in E(G)$ that is not a loop, then by Deletion-Contraction lemma

$$P_G(H) = P_{G/e}(H) - P_{G-e}(H)$$

By induction, there exists polynomials $P_{G/e}(|H|)$ and $P_{G-e}(|H|)$ with the desired property. Hence $P_G(|H|) := P_{G/e}(|H|) - P_{G-e}(|H|)$ is the desired polynomial.

So we may assume every $e \in E(G)$ is a loop.



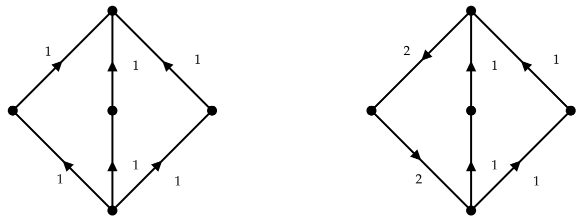
Any assignment of nonzero values (in H) to $\vec{E}(G)$ is an H-flow of G. So $P_G(H) = (|H| - 1)^{|E(G)|}$ and so is a polynomial in $|H|$ as desired. □

k-flow

Let $k \geq 2$ be an integer. A k-**flow** is a function $f : \vec{E}(G) \to \{1, \ldots, k-1\}$ such that

- $f\big((e,x,y)\big) = -f\big((e,y,x)\big)$ for all $(e,x,y) \in \vec{E}(G)$, and
- $\sum\limits_{e=xy\sim x} f\big((e,x,y)\big) = 0$ for all $x \in V(G)$.

Are these related to $\mathbb{Z}_k$-flows?



$\mathbb{Z}_3$-flow vs. 3-flow

Theorem 5.5: Tutte (1950), k-flows and $\mathbb{Z}_k$-flows

Let $k \geq 2$ be an integer. A graph G has a $\mathbb{Z}_k$ flow if and only if it has a k-flow.

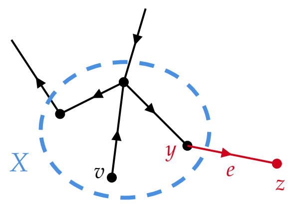
One direction is obvious: k-flow $\implies$ $\mathbb{Z}_k$-flow. It's hard to prove the converse: If there exists an assignment of values in $[k-1]$ to $E(G)$ such that the sum at every vertex is $0 (\bmod k)$, then there exists such an assignment where the sum at every vertex is actually 0.

Proof:

Let f be a $\mathbb{Z}_k$-flow of G such that $\sum\limits_{v \in V(G)} |f(v, V(G))|$ is minimized.

Case 1: $f(v, V(G)) = 0$ for all $v \in V(G)$. Then f is a k-flow as desired.

Case 2: there exists $v \in V(G)$ such that $f(v, V(G)) > 0$. Let X be the set of vertices reachable from v by directed (positive) paths. We claim that there exists $x \in X$ such that $f(x, V(G)) < 0$.



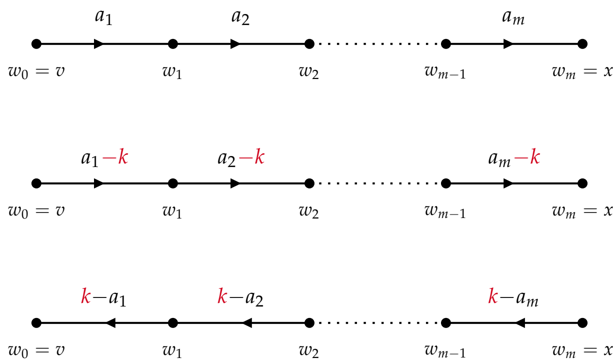
Proof:

Suppose not. Hence $f(X, V(G)) > 0$; yet $f(X, X) = 0$ Thus

$$f(X, V(G) \setminus X) = f(X, V(G)) - f(X, X) > 0$$

So there exists $(e, y, z) \in \vec{E}(X, V(G) \setminus X)$ such that $f\big((e,y,z)\big) > 0$. But then z is reachable from v by a directed path, a contradiction. $\square$

Let $P = w_0 \ldots w_m$ be the directed path from $w_0 = v$ to $w_m = x$.



Now construct a new k-flow f' by assigning

$$f'(w_i w_{i+1}, w_i, w_{i+1}) := f(w_i w_{i+1}, w_i, w_{i+1}) - k$$

Then

$$f'(w_i, V(G)) = \begin{cases} f(w_0, V(G)) - k & i = 0 \\ f(w_i, V(G)) & 1 \le i \le k-1 \\ f(w_m, V(G)) + k & i = m \end{cases}$$

Hence f' is a $\mathbb{Z}_k$-flow such that

$$\sum_{u \in V(G)} |f'(u, V(G)) < \sum_{u \in V(G)} |f(u, V(G))|$$

contradicting the minimality of f. □

Theorem (Tutte 1954)

A graph has an H-flow if and only if it has an H'-flow for all groups H' of order $|H|$.

Theorem (Tutte 1950)

A graph G has a $\mathbb{Z}_k$-flow if and only if it has a k-flow.

Note that if a graph has a k-flow then it has a k'-flow for all $k' \ge k$.

flow number

The **flow number** of a graph G, denoted $\varphi(G)$, is the minimum number k such that G has a k-flow.

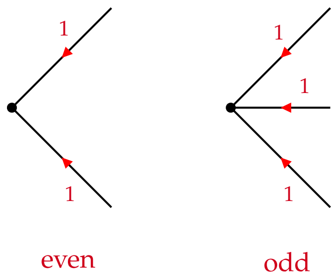
Note that if a graph has a bridge (cut-edge), it admits no flows.

5.2 Small flows

Theorem 5.6

A graph has a 2-flow if and only if all its degrees are even.

Proof:
By Tutte, a graph has a 2-flow if and only if it has a $\mathbb{Z}_2$-flow. A vertex satisfies Kirchoff's law for a $\mathbb{Z}_2$-flow if and only if it has even degree.

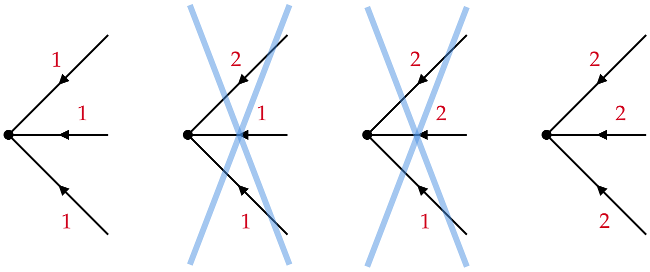


□

Theorem 5.7

The cubic graph has a 3-flow if and only if it is bipartite.

Proof:
A degree 3 vertex satisfies Kirchoff's law for a $\mathbb{Z}_3$-flow if and only if all its incident edges have the same value.



There are two such nonzero values in $\mathbb{Z}_3$: 1 and 2. Hence if a cubic graph has a $\mathbb{Z}_3$-flow, then G is bipartite.

Conversely if $G = (A, B)$ is bipartite, direct edges from A to B with value 1. □

Theorem 5.8: Jaeger (1979)

Every 4-edge connected graph has a 4-flow.

Proof:
By Tutte, it suffices to construct a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow.

Since G is 4-edge-connected, by a Theorem of Nash-Williams, there exist two edge-disjoint spanning trees T_1, T_2 of G. For all $e \notin E(T_1)$, flow value of $(1,0)$ on its fundamental cycle C_{e,T_1}. Let

$$F := \left\{ e' \in E(T_1) : \sum_{e \notin E(T_1)} f_{C_{e,T_1}}(e') = (0,0) \right\}$$

For all $e' \in F$, flow value of $(0,1)$ on its fundamental cycle C_{e',T_2}. □

Theorem 5.9

A graph has a 4-flow if and only if it is the union of two even subgraphs.

Proof:
It follows since by Tutte G has 4-flow if and only if it has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. □

Theorem 5.10

A cubic graph has a 4-flow if and only if it is 3-edge-colorable.

Proof:
A degree 3 vertex satisfies Kirchoff's law for a $\mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if its incident edges have different values. Namely $(1,0), (0,1)$, and $(1,1)$ (three different colors). $\square$

5.3 Flow-Coloring Duality

planar dual

Let G be a plane graph. Then the **dual** of G, denoted by G^*, is the plane graph where

- $V(G^*) = F(G)$,
- $E(G^*) = \{ff' : \exists e \in E(G) \text{ s.t. } f \sim e \sim f'\}$,
- $F(G^*) = V(G)$.

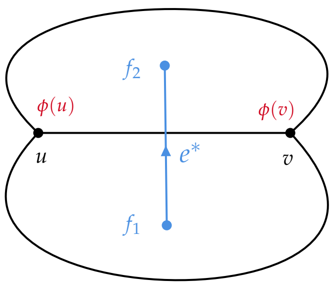
Theorem 5.11: Tutte (1954)

If G is a plane graph, then $\chi(G) = \varphi(G^*)$.

It suffices to prove for connected graphs.

From coloring to flows

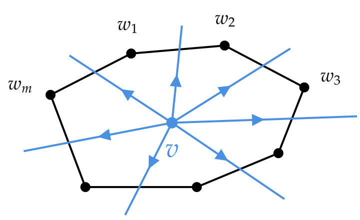
Let ϕ be a k-coloring of G. Define f on $E(G^*)$ by letting $f(e^*) := \phi(u) - \phi(v)$, where u is the left end of e as viewed from the direction of e^*.



Note $|f(e^*)| \leq k - 1$ since $\phi(u), \phi(v) \in [k]$. As ϕ is a coloring, g is nowhere-zero.

We claim f satisfies Kirchoff's law.

Proof:
Let $v \in V(G^*)$, i.e., v is a face of G. Let $w_1 w_2 \ldots w_m$ be the boundary walk of v in G.



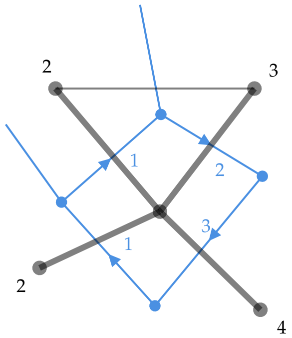
Then

$$\sum_{e^* \sim v} f(e^*) = (\phi(w_1) - \phi(w_2)) + (\phi(w_2) - \phi(w_3)) + \cdots (\phi(w_m) - \phi(w_1)) = 0$$

□

From flows to colorings

Let f be a k-flow of G^*. Let $v \in V(G)$ and T a spanning tree of G. Let $\phi(v) := 1$. Now invert the previous procedure to define a color for each vertex in $V(G)$.



Namely by using the values of $f(e^*)$ for each edge $e \in E(T)$.

We claim that ϕ is a coloring. Namely for all $e = xy \in E(G)$, $\phi(x) \neq \phi(y)$.

Proof:
If $e \in E(T)$, this follows since f is nowhere-zero.

If $e = xy \notin E(T)$, consider the fundamental cycle C_e. Now C_e corresponds to a cut $(X, V(G) \setminus X)$ of G^*. But then

$$0 = f(X, V(G) \setminus X) = f(e^*) + \sum_{e' \in C_e \setminus \{e\}} f((e')^*) = f(e^*) + \phi(x) - \phi(y)$$

Since $f(e^*) \neq 0$, $\phi(x) \neq \phi(y)$. □

Then we have planar coloring theorems, flow version.

Dual of Four Color Theorem

Every bridgeless planar graph has a 4-flow.

Dual of Grötzsch's Theorem

Every planar 4-edge-connected graph has 3-flow.

Could these flow theorems hold for general graphs?

Note the Petersen graph has no 4-flow but does have a 5-flow.

Tutte's Flow Conjectures:

- 5-flow Conjecture (Tutte 1954): Every bridgeless has a 5-flow.
- 4-Flow Conjecture (Tutte 1966): Every bridgeless Petersen-minor-free graph has a 4-flow.
- 3-Flow Conjecture (Tutte 1972): Every 4-edge-connected graph has a 3-flow.

5-Flow Conjecture

Theorem (Jaeger 1979)

Every bridgeless graph has an 8-flow.

The proof constructs a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow.

Theorem (Seymour 1981)

Every bridgeless graph has a 6-flow.

The proof constructs a $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow.

4-Flow Conjecture

For planar graphs, 4-Flow Conjecture is equivalent to the Four Color Theorem.

Theorem (Roberston,Sanders, Seymour, Thomas 1998-2020+)

True for cubic graphs.

Note even this generalizes the Four Color Theorem.

3-Flow Conjecture

Kochol (2001) equivalent to prove for 5-edge-connected graphs.

Theorem (Thomassen 2012)

Every 8-edge-connected graph has a 3-flow.

Theorem (Lovász, Thomassen, Wu and Zhang 2013)

Every 6-edge-connected graph has a 3-flow.

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