Algebraic Graph Theory

CO 444

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LaTeXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of CO 444 during Winter 2021 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

This set of notes is incomplete. For the rest of the contents, please refer to the textbook *Algebraic Graph Theory* by Chris Godsil, Gordon Royle.

For any questions, send me an email via https://notes.sibeliusp.com/contact.

You can find my notes for other courses on https://notes.sibeliusp.com/.

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Graphs

1.1 Graphs

- $X = (V, E)$ where V is set of vertices and E is set of edges.
- neighbours, adjacency, incidence.
- graph isomorphism. $X \cong Y$ if there exists a bijection $f : V(X) \to V(Y)$ such that $uv \in E(x) \iff f(u)f(v) \in E(Y)$.
- complete graphs, (complete) bipartite graphs, empty graphs, the null graph, multigraphs, simple graphs, directed graphs, finite graphs, infinite graphs.

1.2 Subgraphs

subgraph

Y is a **subgraph** of X if $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$.

Spanning subgraph $V(Y) = V(X)$

Induced subgraph $uv \in E(Y) \iff uv \in E(X)$ for all $u, v \in V(Y)$.

cliques, independent sets, paths, cycles, spanning trees.

1.3 Automorphism

automorphism

An isomorphism $f : X \to X$ is an **automorphism** of X.

$\operatorname{Aut} X = \{f : f \text{ automorphism of } X\}$

Let $\operatorname{Sym}(V(X))$ denote the symmetric group of all permutations on $V(X) \implies \operatorname{Aut}(X) \subseteq \operatorname{Sym}(V(X))$.

Let $\operatorname{Sym}(n)$ denote the symmetric group of all permutations on $[n]$.

Proposition 1.1

$\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Sym}(V(X))$.

For $g \in \mathrm{Sym}(V(X))$, and $v \in V(X)$. Let v^g denote $g(v)$, S^g denote $\{g(v) : v \in S\}$.

If $Y \subseteq X$ and $g \in \mathrm{Aut}(X)$. Y^g is a graph defined by

$$V(Y^g) = V(Y)^g, \qquad E(Y^g) = \{g(u)g(v) : uv \in E(Y)\}$$

$Y^g \subseteq X$, and $Y^g \cong Y$.

A few basic properties of $\mathrm{Aut}(X)$

Lemma 1.2: automorphism preserves degrees

Let $v \in V(X)$ and $g \in \mathrm{Aut}(X)$, then $\deg(v) = \deg(g(v))/$

Proof:
Let $Y(v)$ be the subgraph of X induced by $\{v\} \cup N(v)$. Then $Y(v) \cong Y^g = Y(g(v))$. So $\deg(v) = \deg(g(v))$. □

Let $d(x, y)$ be the length of shortest path from x to y.

Lemma 1.3: automorphism preserves graph distance

Let $u, v \in V(X)$ and $g \in \mathrm{Aut}(X)$. Then $d(u, v) = d(u^g, v^g)$.

Proof:
Easy to see that a shortest path from x to y is mapped to a shortest path from $g(x)$ to $g(y)$. □

We will learn more properties of $\mathrm{Aut}(X)$.

1.4 Homomorphisms

homomorphism

X, Y are graphs. $f : V(X) \to V(Y)$ is called a **homomorphism** if $x \sim y$ in $X \implies f(x) \sim f(y)$ in Y.

Examples skipped.

Recall graph coloring and the chromatic number of a graph.

Lemma 1.4

$\chi(X) = \min\{r \in \mathbb{N} : \exists f \text{ homomorphism from } X \text{ to } K_r\}$.

Proof:
Let $k = \chi(X)$. Let g be a k-coloring of X. Then the map sending $v \in V(X)$ to $g(v)$ is a homomorphism from X to K_k. Hence $k \geq \min\{r \in \mathbb{N} : \exists f \text{ homomorphism } X \to K_r\}$.

Suppose f is a homomorphism from X to K_r. Then $f^{-1}(i)$ induces an independent set of X for all

$i \in [r]$. So X has an r-coloring, i.e., $k \leq r$. □

Remark:
The set of homomorphisms from X to K_r is the set of r-colorings with colours $[r]$.

retraction

A **retraction** is a homomorphism $f : V(X) \to V(Y)$ such that

- Y is a subgraph of X,
- $f \mid Y$ (restriction of f to $V(Y)$) is the identity map.

If such a retraction from X to Y exists, we call Y a **retract** of X.

Examples skipped.

1.5 Graph examples

1.5.1 Circulant graphs

C_n

$$V(C_n) = \{0, 1, 2, \ldots, n-1\}$$
$$E(C_n) = \{uv : u - v \equiv \pm 1 \pmod{n}\}$$

Then what is $\mathrm{Aut}(C_n)$?

Let $g = (1, 2, \ldots, n-1, 0)$, $g \in \mathrm{Aut}(C_n)$. R a subgroup of $\mathrm{Aut}(C_n)$, where $R = \{g^m : 0 \leq m \leq n-1\}$.

Let $h \in \mathrm{Sym}(V(C_n))$ such that $h(i) \equiv -i \pmod{n}$. $h \in \mathrm{Aut}(C_n)$.

$\implies hR$: A coset of R different from R.

$\implies |\mathrm{Aut}(C_n)| \geq 2|R| = 2n$.

To generalize, we get circulant graphs. $C \subseteq \mathbb{Z}_n \setminus \{0\}$. Closed under inverse. $c \in C \implies -c \in C$.

$X = X(\mathbb{Z}_n, C)$ where $V(X) = \mathbb{Z}_n$ and $E(X) = \{ij : i - j \in C\}$,

$g, h \in \mathrm{Aut}(X)$ as in the case for C_n, this implies $|\mathrm{Aut}(X)| \geq 2n$.

Further generalization: Cayley graphs.

1.6 Johnson graphs

Johnson graph

Let $v \geq k \geq i$, $J = J(v, k, i)$.

$$V(J) = \{S \subseteq [v] : |S| = k\} \qquad E(J) = \{\{S, T\} : |S \cap T| = i\}$$

$J(v, k, i)$ is d-regular with $d = \binom{k}{i}\binom{v-k}{k-i}$.

Lemma 1.5

$J(v,k,i) \cong J(v,v-k,v-2k+i)$.

Proof:
$f(S) = \bar{S}$ is an isomorphism. □

$J(v,k,0)$: Kneser graph. $J(5,2,0)$: Peterson graph.

Lemma 1.6

$\mathrm{Aut}(J(v,k,i))$ contains a subgroup isomorphic to $\mathrm{Sym}(v)$.

Proof:
Let $g \in \mathrm{Sym}(v)$. Let $\sigma_g : V(J(v,k,i)) \to V(J(v,k,i))$, $S \to S^g$.

It's easy to see that $|S \cap T| = |S^g \cap T^g|$. Then $\sigma_g \in \mathrm{Aut}(J(v,k,i))$. Then $\{\sigma_g : g \in \mathrm{Sym}(v)\} \subseteq \mathrm{Aut}(J(v,k,i))$ and $\{\sigma_g : g \in \mathrm{Sym}(v)\} \cong \mathrm{Sym}(v)$. □

Remark:
$\mathrm{Aut}(J(v,k,i))$ is usually isomorphic to $\mathrm{Sym}(v)$.

1.6.1 Line graphs

$L(X)$

$$V(L(X)) = E(X)$$
$$E(L(X)) = \{\{e,f\} : e \cap f \neq \varnothing\}$$

$X \cong Y \implies L(X) \cong L(Y)$. Converse false. The converse is true however if the minimum degrees of X and Y are ≥ 4.