Partial Differential Equations 2

AMATH 453

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Preface

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Sibelius Peng

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Waves and Diffusions

1.1 The wave equation

We already know the wave equation ($c > 0$):

$$u_{tt} - c^2 u_{xx} = 0, \qquad -\infty < x < \infty,$$

and the general solution is of the form

$$u(x,t) = f(x+ct) + g(x-ct).$$

With initial conditions imposed, we have the IVP

$$u_{tt} - c^2 u_{xx} = 0, \qquad \begin{cases} u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x). \end{cases}$$

The solution to IVP is then

$$u(x) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

To interpret the integral, we can let $\psi(x) = \mu'(x)$, then the integral becomes

$$\int_{x-ct}^{x+ct} \psi(s)\, ds = \mu(x+ct) - \mu(x-ct).$$

1.2 Conservation laws

Given a wave equation, we multiply by u_t:

$$\begin{gathered} u_t u_{tt} - c^2 u_t u_{xx} = 0 \\ \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2\right) - c^2\left[\frac{\partial}{\partial x}(u_t u_x) - u_{tx}u_x\right] = 0 \\ \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) - \frac{\partial}{\partial x}\left(c^2 u_t u_x\right) = 0 \end{gathered}$$

Then the conservation law states that

$$\frac{\partial R}{\partial t} + \frac{\partial F}{\partial x} = 0\,,$$

where $R \in (-\infty, +\infty)$, and $F \to 0$ with $x \to \pm\infty$.

1.3 The Diffusion Equation & Maximum principle

The diffusion equation is given by

$$u_t = ku_{xx}, \qquad -\infty < x < \infty$$

with diffusion constant $k > 0$.

We define

$$\begin{aligned} R &= (a,b) \times (0,\infty) \\ R_T &= (a,b) \times (0,T] \\ \overline{R_T} &= [a,b] \times [0,T] \\ C_T &= \{a \le x \le b, t = 0\} \cup \{a, 0 \le t \le T\} \cup \{b, 0 \le t \le T\} \end{aligned}$$

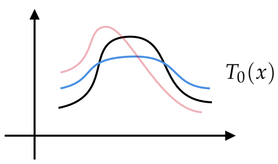
Theorem 1.1: Maximum principle

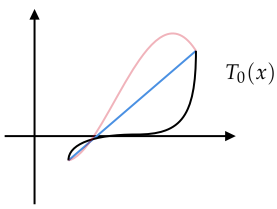
If $u \in C(\overline{R_T}) \cap C^2(R_T)$ is a solution of the diffusion equation, then $u(x,t) \le \max_{C_T}\{u\}$ for all $(x,t) \in R_T, T > 0$. Here C_T is called the parabolic boundary of R_T.

Remark:

1. We can replace $u_t - ku_{xx} = 0$ with $u_t - ku_{xx} \le 0$.
2. A stronger version of the theorem exists which says that $u(x,t) < \max_{C_T}\{u\}$ unless u is constant.
3. Same result applies to the minimum of u by replacing u with $-u$. However, in this case, (1) doesn't apply. Now we need $u_t - ku_{xx} \ge 0$.

Here are some intuitions. Consider a rod lying on $[a,b]$ with initial non-constant temperature $T_0(x)$. Then as time goes, only blue T is possible, not red T.





Proof:
Let $M = \max_{C_T} u$. Note that M exists since u is continuous on C_T, and C_T is a closed boundary. We need to show that $u \le M$ on $\overline{R_T}$.

Let

$$v(x,t) = u(x,t) + \epsilon x^2, \quad \epsilon > 0$$

Let $r = \max\{|a|,|b|\}$. Then $v(x,t) \le M + \epsilon r^2$ on C_T. Now we prove that $v \le M + \epsilon r^2$ on R_T.

On R_T, we have

$$u = v - \epsilon x^2 \le M + \epsilon(r^2 - x^2)$$

Now if we take the derivative,

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon = -2k\epsilon < 0 \tag{*}$$

(i) Suppose $v(x,t)$ has a maximum at an interior point (x_0,t_0), i.e., $(x_0,t_0) \in (a,b) \times (0,T)$. Then

$v_t(x_0,t_0) = 0$. Moreover, $v_{xx}(x_0,t_0) \le 0$. Then

$$v_t(x_0,t_0) - kv_{xx}(x_0,t_0) = -kv_{xx}(x_0,t_0) \ge 0$$

contradicting (*), thus there are no interior max.

(ii) Suppose $v(x,t)$ has a maximum at an interior point of the upper boundary. $v_t(x_0,T) \ge 0$. Then

$$v_t(x_0,t_0) - kv_{xx}(x_0,t_0) \ge 0$$

contradicting (*), thus there are no maximum along the upper boundary.

But v is continuous on $\overline{R_T}$, thus it has a maximum value which we now know must occur on C_T. Hence $v \le M + \epsilon r^2$ on $\overline{R_T}$. Letting $\epsilon \to 0$, we have $u \le M$ on R_T. □

1.4 Uniqueness of the Dirichlet Problem

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad a < x < b, 0 < t < \infty \\ u(x,0) &= \phi(x) \\ u(a,t) &= g(t) \\ u(b,t) &= h(t) \end{aligned} \tag{1.1}$$

Theorem 1.2

The solution of (1.1) is unique.

Proof:
Suppose there are two solutions $u_1(x,t)$ and $u_2(x,t)$. Let $w(x,t) = u_1 - u_2$. Now we calculate

$$\begin{aligned} w_t - kw_{xx} &= (u_{1t} - ku_{1xx}) - (u_{2t} - u_{2xx}) = f - f = 0 \\ w(x,0) &= u_1(x,0) - u_2(x,0) = \phi - \phi = 0 \\ w(a,t) &= w(b,t) = 0 \end{aligned}$$

By maximum principle, we have $w \le 0$ on the boundary, and my minimum principle, $w \ge 0$, since $\max_{C_T}\{w\} = \min_{C_T}\{w\} = 0$. Then we conclude that $w \equiv 0$. □

Now we present a second proof using energy method:

Proof:
Given $w_t - kw_{xx} = 0$, multiply both sides by w:

$$0 = ww_t - kww_{xx} = \frac{\partial}{\partial t}\left(\frac{1}{2}w^2\right) - k\frac{\partial}{\partial x}(ww_x) + kw_x^2$$

If we integrate both sides,

$$\frac{d}{dt}\int_a^b \frac{1}{2}w^2 \,\mathrm{d}x = k\int_a^b (ww_x)_x \,\mathrm{d}x - k\int_a^b w_x^2 \,\mathrm{d}x = kww_x\Big|_a^b - k\int_a^b w_x^2 \,\mathrm{d}x$$

Thus

$$\frac{d}{dt}\int_a^b \frac{1}{2}w^2 \,\mathrm{d}x = -k\int_a^b w_x^2 \,\mathrm{d}x$$

Then

$$\int_a^b \frac{1}{2}w^2 \,\mathrm{d}x = 0 \qquad \text{for all the time}$$

Then $w \equiv 0$ on $a \le x \le b, 0 \le t \le T$. □

Now let's examine stability. Consider

$$u_t - ku_{xx} = 0$$
$$u(a,t) = u(b,t) = 0$$

and let $u_j(x,t)$ be the solution for $u(x,0) = \phi_j(x)$ for $j = 1,2$.

Let $w = u_1 - u_2$. Proceeding as before (energy method) we have

$$\int_a^b (u_1 - u_2)^2 \, \mathrm{d}x \le \int_a^b (\phi_1 - \phi_2)^2 \, \mathrm{d}x$$

This tells us $\|u_1 - u_2\|_2 \to 0$ as $\|\phi_1 - \phi_2\|_2 \to 0$. This is called **stability in the square integrable sense**.

Alternatively, by maximum principle,

$$\max |u_1 - u_2| \le \max |\phi_1 - \phi_2|$$

using maximum & minimum principle, i.e.,

$$\max\{u_1 - u_2\} \le \max\{\phi_1 - \phi_2\}$$
$$\min\{u_1 - u_2\} \ge \min\{\phi_1 - \phi_2\}$$

This is called **stability in the uniform sense**.

1.5 Diffusion on the Whole Line

Consider the initial value problem

$$u_t - ku_{xx} = 0 \qquad \text{on } -\infty < x < \infty, \quad 0 < t < \infty \tag{1.2}$$
$$u(x,0) = \phi(x) \tag{1.3}$$

If $s(x,t)$ is a solution of (1.2), then so is

$$u(x,t) = \int_{-\infty}^{\infty} s(x-y,t) g(y) \, \mathrm{d}y \tag{1.4}$$

for any function $g(y)$. We can find u_t, u_x, u_{xx} and take it into (1.2):

$$u_t - ku_{xx} = \int_{-\infty}^{\infty} \Big[s_t(x-y,t) - ks_{xx}(x-y,t) \Big] g(y) \, \mathrm{d}y = 0$$

So we now find a solution of (1.2) with the property that $s(x,0) = \delta(x)$, i.e., solve

$$s_t - ks_{xx} = 0$$
$$s(x,0) = \delta(x)$$

To do this, consider the problem:

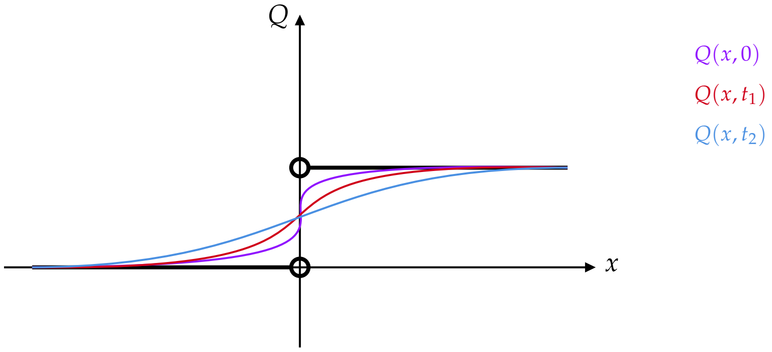
$$v_t - kv_{xx} = 0$$
$$v(x,0) = v_0 H(x) \tag{1.5}$$
$$H = \text{Heaviside function}$$

v_0 carries the dimension of v, thus $H(x)$ is dimensionless.

Similarity solution of (1.5)

Let $Q = \frac{v}{v_0}$ which is dimensionless, then the original problem gets transformed to

$$Q_t = kQ_{xx}$$
$$Q(x,0) = H(x)$$



The solution can only be a function of x, t and k: $Q = F(x,t,k)$. Then we can apply dimensionless analysis. This means Q can only depend on dimensionless combinations of x, t and k. We have

$$[x] = L$$
$$[t] = T$$
$$[k] = \frac{L^2}{T}$$

Then

$$[x^a t^b k^c] = L^a T^b \frac{L^{2c}}{T^c} \implies b = c,\ 2c = -a$$

This tells us

$$Q = f(\theta) \qquad \text{where } \theta = \frac{x}{\sqrt{kt}}$$

By chain rule, we have

$$Q_t = f'(\theta)\cdot\theta_t = -\frac{1}{2}\frac{\theta}{t}f'(\theta)$$
$$Q_x = f'(\theta)\cdot\theta_x = \frac{1}{\sqrt{kt}}f'(\theta)$$
$$Q_{xx} = \frac{1}{kt}f''(\theta)$$

Then

$$Q_t - kQ_{xx} = -\frac{\theta}{2t}f' - \frac{k}{kt}f'' = 0$$
$$f''(\theta) = -\frac{1}{2}\theta f'(\theta)$$
$$f'(\theta) = Ae^{-\frac{\theta^2}{4}}$$
$$f(\theta) = A\int_{-\infty}^{\theta} e^{-s^2/4}\,\mathrm{d}s + C$$

As $x \to +\infty$, $\theta \to +\infty$, and $Q(x,t) = f(\theta) \to 1$. Then $\lim_{\theta\to+\infty} f(\theta) = 1$.
As $x \to -\infty$, $\theta \to -\infty$ and $Q(x,t) = f(\theta) \to 0$, $\lim_{\theta\to-\infty} f(\theta) = 0$.

Therefore, C must be 0, and $A\int_{-\infty}^{\infty} e^{-s^2/4}\,\mathrm{d}s = 1$. Using the change of variable $\eta = \frac{s}{2}$:

$$\int_{-\infty}^{\theta} e^{-s^2/4}\,\mathrm{d}s = 2\int_{-\infty}^{\theta/2} e^{-\eta^2}\,\mathrm{d}\eta = 2\int_{-\infty}^{x/\sqrt{4kt}} e^{-\eta^2}\,\mathrm{d}\eta$$

So if we take $\theta = \dfrac{x}{\sqrt{4kt}}$ at the beginning, we get $\tilde{A} = 2A$ and

$$\tilde{A} \int_{-\infty}^{\infty} e^{-s^2} \, \mathrm{d}s = 1 \implies \tilde{A} = \frac{1}{\sqrt{\pi}}$$

Thus we get

$$Q = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4kt}} e^{-s^2} \, \mathrm{d}s$$

Note that for $x > 0$, as $t \to 0^+$, $\dfrac{x}{\sqrt{4kt}} \to +\infty$ and $Q(x,t) \to \dfrac{1}{\sqrt{\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-s^2} \, \mathrm{d}s = 1$.

And for $x < 0$ as $t \to 0^+$, $Q \to 0$. The reason for the name "similarity solution" is because the curve is being stretched over time.

$s(x,t)$ has many names: source function (not a great name), Green's function, fundamental solution, propagator of the diffusion equation, diffusion kernel...

Consider a diffusion equation with initial condition

$$\begin{aligned} u_t + ku_{xx} &= 0 \\ u(x,0) &= \delta(x) \end{aligned}$$

The solution is Gaussian

$$u = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

For any $t > 0$, u is non-zero. It gets instantaneously non-zero everywhere.

2

Reflections and Sources

2.1 Diffusion on the Half-Line

We will start with diffusion on the half line Dirichlet problem.

$$
\begin{aligned}
v_t - kv_{xx} &= 0 \qquad 0 < x < \infty,\, 0 < t < \infty \\
v(x,0) &= \phi(x) \\
v(0,t) &= 0 \quad \text{for } t > 0
\end{aligned}
$$

Let

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases}$$

and solve

$$
\begin{aligned}
u_t + ku_{xx} &= 0 \qquad \text{on } -\infty < x < \infty \\
u(x,0) &= \phi_{odd}(x)
\end{aligned}
$$

Then $v(x,t)$ is restriction of u to $x > 0$. From an earlier result

$$u(x,t) = \int_{-\infty}^{\infty} s(x-y,t)\phi_{odd}(y)\, \mathrm{d}y$$

where

$$s(x,t) = \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4\pi kt}}$$

Claim From the property of s and ϕ_{odd}, we can show that $u(x,t)$ is an odd function of x. Thus $u(0,t) = 0$.

Now we see that

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{0} s(x-y,t)[-\phi(-y)]\, \mathrm{d}y + \int_0^{\infty} s(x-y,t)\phi(y)\, \mathrm{d}y \\
&= \int_{\infty}^{0} s(x+y,t)\phi(y)\, \mathrm{d}y + \int_0^{\infty} s(x-y,t)\phi(y)\, \mathrm{d}y \qquad \text{let } y = -y \\
&= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y)\, \mathrm{d}y \qquad (2.1)
\end{aligned}
$$

Example:

$$\begin{aligned} v_t - kv_{xx} &= 0 \qquad 0 < x < \infty \\ v(x,0) &= 1 \qquad x > 0 \\ v(0,t) &= 0 \end{aligned}$$

Then $\phi_{odd} = -1 + 2H(x)$.

Recall the solution of

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x,0) &= H(x) \end{aligned} \tag{2.2}$$

is

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4k\pi}} e^{-s^2}\, \mathrm{d}s$$

Let $u(x,t) = -1 + 2q(x,t)$. Then $q(x)$ is the solution to (2.2). Hence we have

$$u = -1 + \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4k\pi t}} e^{-s^2}\, \mathrm{d}s = \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

Another way to solve is to use (2.1).

Consider Neumann Boundary condition ($0 < x < \infty$):

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x,0) &= \phi(x) \\ u_x(0,t) &= 0 \end{aligned}$$

We can let

$$\phi_{even} = \begin{cases} \phi(x) & x > 0 \\ \phi(-x) & x < 0 \end{cases}$$

and solve

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x,0) &= \phi_{even} \end{aligned}$$

With some algebra, we get

$$u = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y)\, \mathrm{d}y$$

2.2 Reflections of Waves

Dirichlet Problem on the half line

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0 \qquad 0 < x < \infty \\ v(x,0) &= \phi(x) \\ v_t(x,0) &= \psi(x) \\ v(0,t) &= 0 \end{aligned}$$

The idea is $u(-x,t) = -u(x,t)$, then $u = 0$ at $x = 0$. So consider an odd reflection about $x = 0$:

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases} \qquad \psi_{odd} = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0 \end{cases}$$

We know that the solution of $(-\infty < x < \infty)$

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(x,0) &= \phi_{odd}(x) \\ u_t(x,0) &= \psi_{odd}(x) \end{aligned}$$

is

$$u(x,t) = \frac{1}{2}\Big[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\ \mathrm{d}y$$

Note that $(t > 0)$

$$u(0,t) = \frac{1}{2}\Big[\phi_{odd}(ct) + \phi_{odd}(-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\ \mathrm{d}y = 0$$

which satisfies the initial condition.

3 cases of the solution

(a) $x > c|t|$, then $x+ct > 0, x-ct > 0$, then the solution $(t > 0)$ becomes

$$u(x,t) = \frac{1}{2}\Big[\phi(x+ct) + \phi(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\ \mathrm{d}y$$

(b) Consider $0 < x < ct$, $t > 0$, we have $x-ct < 0, x+ct > 0$. Then

$$\begin{aligned} \phi_{odd}(x-ct) &= -\phi(-x+ct) \\ \phi_{odd}(x+ct) &= \phi(x+ct) \end{aligned}$$

and

$$\begin{aligned} \int_{x-ct}^{x+ct} \psi_{odd}(y)\ \mathrm{d}y &= \int_{x-ct}^{0} [-\psi(-y)]\ \mathrm{d}y + \int_0^{x+ct} \psi(y)\ \mathrm{d}y \\ &= -\int_0^{-x+ct} \psi(y)\ \mathrm{d}y + \int_0^{x+ct} \psi(y)\ \mathrm{d}y \\ &= \int_{-(x-ct)}^{x+ct} \psi(y)\ \mathrm{d}y \end{aligned}$$

Therefore

$$u = \frac{1}{2}\Big[\phi(x+ct) - \phi(-(x-ct))\Big] + \frac{1}{2c}\int_{-(x-ct)}^{x+ct} \psi(y)\ \mathrm{d}y$$

2.3 Diffusion with a Source

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad -\infty < x < \infty \\ u(x,0) &= \phi(x) \qquad\quad 0 < t < \infty \end{aligned}$$

We can solve

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \\ u(x,0) &= 0 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x,0) &= \phi(x) \end{aligned}$$

and sum to get the solution.

Duhamel's Principle for first order linear ODEs

The solution of

$$\begin{aligned} y' + ay &= F(t) \qquad t > 0, \quad a \text{ constant} \\ y(0) &= 0 \end{aligned}$$

is given by

$$y(t) = \int_0^t w(t-s;s)\ \mathrm{d}s$$

where $w(t;s)$ is the solution of

$$\begin{aligned} w_t(t;s) + aw(t;s) &= 0 \\ w(0;s) &= F(s) \end{aligned}$$

Proof:

$$\frac{d}{dt}(e^{at}y) = e^{at}F(t) = e^{at}y = \int_0^t e^{as}F(s)\ \mathrm{d}s$$

Then

$$y = \int_0^t e^{a(s-t)}F(s)\ \mathrm{d}s$$

Using initial condition $y(0) = 0$ and $w(t,s) = F(s)e^{-at}$,

$$w(t-s;s) = F(s)e^{a(s-t)}$$

Thus

$$y(t) = \int_0^t w(t-s;s)\ \mathrm{d}s$$

□

We are now to guess that this works for the diffusion equation, i.e., guess the solution of (2.3) is

$$u(x,t) = \int_0^t w(x,t-s;s)\ \mathrm{d}s$$

where $w(x,t;s)$ is the solution of

$$\begin{aligned} w_t - kw_{xx} &= 0 \\ w(x,0;s) &= f(x,s) \end{aligned}$$

From previous work

$$w = \int_{-\infty}^{\infty} s(x-y,t)f(y,s)\ \mathrm{d}y$$

Then

$$u = \int_0^t \int_{-\infty}^{\infty} S(x-y,t)f(y,s)\ \mathrm{d}y\ \mathrm{d}s \tag{2.4}$$

We need to verify that this is indeed the solution

$$\begin{aligned} u_t &= \int_{-\infty}^{\infty} s(x-y,0)f(y,t)\ \mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} s_t(x-y,t-s)f(y,s)\ \mathrm{d}y\ \mathrm{d}s \\ &= f(x,y) + \int_0^t \int_{-\infty}^{\infty} s_t(x-y,t-s)f(y,s)\ \mathrm{d}y\ \mathrm{d}s \end{aligned}$$

Next

$$u_{xx} = \int_0^t \int_{-\infty}^{\infty} s_{xx}(x-y,t-s)f(y,s)\ \mathrm{d}y\ \mathrm{d}s$$

Then we see that $u_t - ku_{xx} = f(x,t)$ and $u(x,0) = 0$.

Therefore (2.4) is a solution of (2.3). Then add $\int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y$ to add IC $u(x,0) = \phi(x)$.

2.4 Source on a half line

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad 0 < x < \infty, \quad 0 < t < \infty \\ u(x,0) &= \phi(x) \\ u(0,t) &= h(t) \end{aligned}$$

where $h(t)$ is the source on the boundary.

Let $v(x,t) = u(x,t) - h(t)$, then

$$\begin{aligned} v_t - kv_{xx} &= u_t - ku_{xx} - h'(t) = f(x,t) - h'(t) \\ v(x,0) &= \phi(x) - h(0) = \tilde{\phi}(x) \\ v(0,t) &= 0 \end{aligned}$$

Then we can use odd extension and solve

$$\begin{aligned} v_t - kv_{xx} &= \tilde{f}(x,t) := f_{odd} - h'(t) \\ v(x,0) &= \tilde{\phi}_{odd} \end{aligned}$$

Use previous solution and restrict to the positive x-axis to get $v(x,t)$ and then $u(x,t) = v(x,t) + h'(t)$.

Theorem 2.1

Let $\phi(x)$ be a bounded continuous function on $-\infty < x < \infty$. Then

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y \tag{2.5}$$

where

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

defines an C^∞ solution of

$$\begin{aligned} u_t - ku_{xx} &= 0 \qquad -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,0) &= \phi(x) \end{aligned}$$

Proof:

Sub $S(x,t)$ in, we get

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\,\mathrm{d}y$$

We now introduce the change of variable,

$$\frac{x-y}{\sqrt{kt}} = p$$

then

$$y = x - \sqrt{kt}p, \qquad \mathrm{d}y = -\sqrt{kt}\,\mathrm{d}p$$

Then

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x-\sqrt{kt}p)(-\sqrt{kt}\,\mathrm{d}p) \\ &= \frac{1}{\sqrt{4\pi}} \int_{\infty}^{\infty} e^{-p^2/4} \phi(x-\sqrt{kt}p)\,\mathrm{d}p \end{aligned}$$

Thus

$$\begin{aligned} |u(x,t)| &\le \frac{1}{\sqrt{4\pi}} \int_{\infty}^{\infty} e^{-p^2/4} \Big|\phi(x-\sqrt{kt}p)\Big|\,\mathrm{d}p \\ &= \frac{\max|\phi|}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4}\,\mathrm{d}p \\ &= \max|\phi| \end{aligned}$$

Thus (2.5) integral converges absolutely and uniformly.

Formally

$$u_x(x,t) = \infty_{-\infty}^{\infty} \frac{\partial}{\partial x} S(x-y,t)\phi(y)\,\mathrm{d}y$$

and these two are equal of the integral converses absolutely.

Consider

$$\begin{aligned} I(x,t) &= \int_{-\infty}^{\infty} S_x(x-y,t)\phi(y)\,\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left[-\frac{(x-y)}{2kt} e^{-\frac{(x-y)^2}{4kt}}\right] \phi(y)\,\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{-\sqrt{kt}p}{2kt} e^{-p^2/4} \phi(x-\sqrt{kt}p)\sqrt{kt}\,\mathrm{d}p \\ &= -\frac{1}{4\sqrt{\pi k}} \frac{1}{\sqrt{t}} \phi(x-\sqrt{kt}p)\,\mathrm{d}p \end{aligned}$$

Therefore, for C constant

$$|I| \le \frac{C \max|\phi|}{\sqrt{t}} \int_{-\infty}^{\infty} |p| e^{-p^2/4}$$

converges.

Therefore

$$\int_{-\infty}^{\infty} S_x(x-y,t)\phi(y)\,\mathrm{d}y$$

converges absolutely and hence is equal to u_x. Similarly all $\dfrac{\partial^{m+n} u}{\partial t^m \partial x^n}$ exist because they will all be the sum of integrals of the form $A \int_{-\infty}^{\infty} |p^j| e^{-p^2/4}\,\mathrm{d}p$ which converges for all j.

Hence

$$u_t - k u_{xx} = \int_{-\infty}^{\infty} \Big[S_t(x-y,t) - k S_{xx}(x-y,t)\Big] \phi(y)\,\mathrm{d}y = 0$$

since S is a solution of the diffusion equation.

Now we check the initial condition. Since formally $S(x,t)$ does not exist at $t=0$ by "the IC is satisfied" we mean $\lim_{t\to 0^+} u(x,t) = \phi(x)$. Now

$$u(x,t) - \phi(x) = \int_{-\infty}^{\infty} s(x-y,t)[\phi(y)-\phi(x)]\,\mathrm{d}y$$

Using $y = x - \sqrt{kt}p$ as before

$$u(x,t) - \phi(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \Big(\phi(x-\sqrt{kt}p) - \phi(x)\Big)\,\mathrm{d}p$$

If we fix x, $\phi(x)$ is continuous at x, so for $\epsilon > 0$, there exists $\delta > 0$ such that

$$|y - x| < \delta \implies |\phi(x+\delta) - \phi(x) < \frac{\epsilon}{2}$$

$$\begin{aligned} u(x,t) - \phi(x) &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \Big(\phi(x - \sqrt{kt}p) - \phi(x) \Big) \, \mathrm{d}p \\ &= \frac{1}{\sqrt{4\pi}} \int_{|p| < \frac{\delta}{\sqrt{kt}}} e^{-p^2/4} \Big(\underbrace{\phi(x - \sqrt{kt}p) - \phi(x)}_{\substack{\text{abs value } < \epsilon/2 \\ \text{on } |p| < \delta/\sqrt{kt}}} \Big) \, \mathrm{d}p + \frac{1}{\sqrt{4\pi}} \int_{|p| > \frac{\delta}{\sqrt{kt}}} \cdots \, \mathrm{d}p \\ &\le \frac{\epsilon}{2} + \frac{2 \max |\phi|}{\sqrt{4\pi}} \boxed{\int_{|p| > \frac{\delta}{\sqrt{kt}}} e^{-p^2/4} \, \mathrm{d}p} \end{aligned}$$

Note that the boxed integral satisfies

$$\int_{|p| > \frac{\delta}{\sqrt{kt}}} e^{-p^2/4} \, \mathrm{d}p = 2 \int_{-\delta/\sqrt{kt}}^{\infty} e^{-p^2/4} \, \mathrm{d}p \to 0 \qquad \text{as } t \to 0$$

Thus we can take t small enough to make second term $< \epsilon/2$ to get

$$u(x,t) - \phi(x) < \epsilon$$

if t is sufficiently small. □

Theorem 2.2

Let $\phi(x)$ be a bounded *piecewise* continuous function on $-\infty < x < \infty$. Then

$$u(x,t) = \int_{-\infty}^{\infty} S(x - y, t) \phi(y) \, \mathrm{d}y$$

where

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

defines an C^∞ solution of

$$\begin{aligned} u_t - k u_{xx} &= 0 \qquad -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,0) &= \phi(x) \end{aligned}$$

Proof:

Just need to check the initial conditions which we have to interpret as

$$\lim_{t \to 0^+} u(x,t) = \frac{1}{2} \Big(\phi(x^+) + \phi(x^-) \Big)$$

Now

$$\begin{aligned} u(x,t) - \phi(x) =& \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \Big(\overset{\phi(x^+)}{\boxed{\phi(x - \sqrt{kt}p)}} - \phi(x) \Big) \, \mathrm{d}p \\ &+ \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \Big(\underset{\phi(x^-)}{\boxed{\phi(x - \sqrt{kt}p)}} - \phi(x) \Big) \, \mathrm{d}p \end{aligned}$$

□

2.5 Waves with a Source

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t) \qquad -\infty < x < \infty \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned} \tag{2.6}$$

First we find the solution u_1 of

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \qquad -\infty < x < \infty \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned} \tag{2.7}$$

Then we find the solution u_2 of

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t) \qquad -\infty < x < \infty \\ u(x,0) = u_t(x,0) &= 0 \end{aligned} \tag{2.8}$$

Then $u_1 + u_2$ is a solution of (2.6). We can verify as follows

$$(u_1 + u_2)_{tt} - c^2 (u_1 + u_2)_{xx} = \cdots = f(x,t)$$

and so on. We already know the solution to (2.7):

$$u_1 = \frac{1}{2}\Big(\phi(x+ct) + \phi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy$$

Therefore, we just need to solve (2.8).

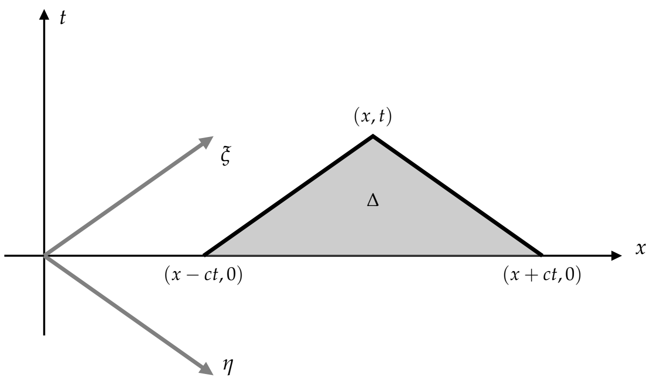
Method 1: Characteristic Coordinates

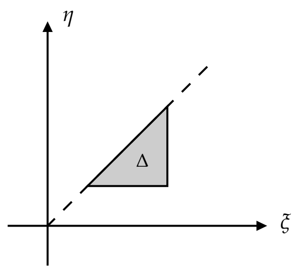
We let

$$\begin{aligned} \eta &= x + ct \\ \xi &= x + ct \end{aligned}$$

In other words, we have

$$x = \frac{\xi + \eta}{2} \qquad t = \frac{\xi - \eta}{2c}$$





Under this transformation

$$\begin{aligned}\frac{\partial}{\partial t}+c\frac{\partial}{\partial x} &= \eta_t\frac{\partial}{\partial \eta}+\xi_t\frac{\partial}{\partial \xi}+\left(\eta_x\frac{\partial}{\partial \eta}+\xi_x\frac{\partial}{\partial \xi}\right)\\ &= (\eta_t+c\eta_x)\frac{\partial}{\partial \eta}+(\xi_t+c\xi_x)\frac{\partial}{\partial \xi}\\ &= (-c+c)\frac{\partial}{\partial \eta}+(c+c)\frac{\partial}{\partial \xi}\\ &= 2c\frac{\partial}{\partial \xi}\end{aligned}$$

Similarly

$$\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}=-2c\frac{\partial}{\partial \eta}$$

Therefore

$$u_{tt}-c^2u_{xx}=f \implies -4c^2u_{\xi\eta}=f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)$$

Then we can let

$$\tilde{u}_{\eta\xi}=-\frac{1}{4c^2}\tilde{f}(\eta,\xi) \tag{2.9}$$

Also we can transform the initial conditions as well:

$$\begin{aligned}u(x,0)=0 &\implies \tilde{u}(\xi,\xi)=0\\ u_t(x,0)=0 &\implies \tilde{u}_\eta=\tilde{u}_\xi \text{ or } \eta=\xi\end{aligned}$$

because

$$\begin{aligned}u_t(x,t) &= \xi_t\frac{\partial\tilde{u}}{\partial \xi}+\eta_t\frac{\partial\tilde{u}}{\partial \eta}\\ &= c\frac{\partial\tilde{u}}{\partial \xi}-c\frac{\partial\tilde{u}}{\partial \eta}\end{aligned}$$

Then we integrate $\tilde{u}_{\xi\eta}$ on characteristic triangle Δ:

$$\begin{aligned}I=\iint_\Delta \tilde{u}_{\xi\eta}\,\mathrm{d}\eta\,\mathrm{d}\xi &= \int_{\xi=\eta_0}^{\xi_0}\int_{\eta=\eta_0}^{\xi} u_{\xi\eta}\,\mathrm{d}\eta\,\mathrm{d}\xi\\ &= \int_{\eta_0}^{\xi_0} u_\xi\Big|_{\eta=\eta_0}^{\eta=\xi}\,\mathrm{d}\xi\\ &= \int_{\eta_0}^{\xi_0}\left[u_\xi(\xi,\xi)-u_\xi(\eta_0,\xi)\right]\mathrm{d}\xi\end{aligned}$$

Consider the function $g(\xi)=u(\xi,\xi)$, then

$$\frac{dg}{d\xi}=2u(\xi,\xi)$$

using the second IC. Then

$$I=\frac{1}{2}g(\xi_0)-\frac{1}{2}f(\eta_0)-u(\eta_0,\xi_0)+u(\eta_0,\eta_0)=-u(\eta_0,\xi_0)$$

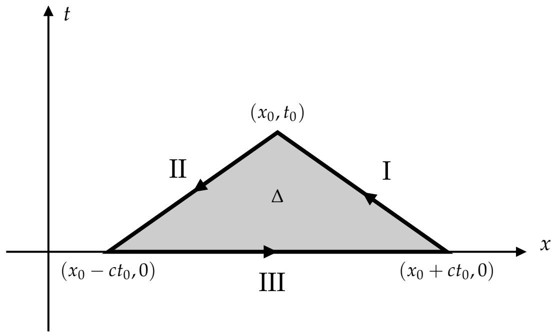
Then we integrate the right side of (2.9) as well:

$$\begin{aligned}-\iint_\Delta &= \frac{1}{4c^2}\iint_\Delta f\\ u(\eta_0,\xi_0) &= \frac{1}{4c^2}\iint_\Delta \tilde{f}(\eta,\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi\end{aligned}$$

Using Jacobian, we have $\mathrm{d}\eta\,\mathrm{d}\xi=2c\,\mathrm{d}x\,\mathrm{d}t$. Then

$$\begin{aligned}u(x,t) &= \frac{1}{2c}\iint_\Delta f(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{2c}\int_0^t\int_{x_0-c(t_0-q)}^{x+c(t_0-q)} f(s,q)\,\mathrm{d}q\,\mathrm{d}s\end{aligned}$$

Method 2: Green's Theorem / Divergence Theorem



Here we have the parametrize curve $\big(x, t(x)\big)$. On I, $\hat{n}\ \mathrm{d}s = (1/c, 1)\ \mathrm{d}x$ and we let

$$t_{\mathrm{I}}(x) = t_0 - \frac{1}{c}(x - x_0)$$

Consider the characteristic triangle in the xt plane.

$$\iint (u_{tt} - c^2 u_{xx})\ \mathrm{d}x\ \mathrm{d}t = \iint f(x,t)\ \mathrm{d}x\ \mathrm{d}t$$

By Divergence theorem,

$$\begin{aligned}\text{LHS} &= \iint \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) \cdot (-c^2 y_x, u_t)\ \mathrm{d}x\ \mathrm{d}t \\ &= \oint (-c^2 u_x, u_t) \cdot \hat{n}\ \mathrm{d}s \\ &= \int_{\mathrm{I}} + \int_{\mathrm{II}} + \int_{\mathrm{III}}\end{aligned}$$

Note that $\int_{\mathrm{III}} = 0$ because $u_t(x,0) = u_x(x,0) = 0$.

Now for side I:

$$\int_{x_0}^{x_0+ct_0} (-c^2 u_x, u_t) \cdot (1/c, 1)\ \mathrm{d}x = \int_{x_0}^{x_0+ct_0} (u_t - cu_x)\ \mathrm{d}x$$

Along I, $u = u(x, t_I(x)) := g_I(x)$, then

$$\begin{aligned}g_I'(x) &= u_x + u_t \frac{dt_I}{dx} \\ &= u_x - \frac{1}{c} u_t \\ &= -\frac{1}{c}(u_t - cu_x)\end{aligned}$$

Therefore

$$\int_{\mathrm{I}} (u_t + cu_x)\ \mathrm{d}x = \int_{x_0}^{x_0+ct_0} -cg_I'\ \mathrm{d}x = -c\big(g(x_0 + ct_0) - g(x_0)\big)$$

Note that

$$\begin{aligned}g(x_0) &= u(x_0, t_0) \\ g(x_0 + ct_0) &= u(x_0 + ct_0, 0) = 0\end{aligned}$$

Therefore

$$\int_{\mathrm{I}} = cu(x_0, t_0)$$

Similarly,

$$\int_{\mathrm{II}} = cu(x_0, t_0)$$

Therefore we have

$$u(x_0, t_0) = \frac{1}{2c} \iint_{\Delta} f \, \mathrm{d}x \, \mathrm{d}t$$

which is identical to the previous result.